# Mapping Class Groups, Homology and Finite Covers of Surfaces

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## Abstract

Let $S$ be an orientable surface of genus $g$ with $n$ punctures, such that $\chi(S) = 2-2g-n < 0$. Let $\psi \in \mathrm{Mod}(S)$ denote an element in its mapping class group. In this thesis, we study the action of $\psi$ on $H_1(\widetilde{S},\mathbb{C})$, where $\widetilde{S}$ varies over the finite covers of $S$ to which $\psi$ lifts.

We first show that if $\psi$ is a nontrivial mapping class then there exists a finite cover $\widetilde{S}$ such that each lift of $\psi$ to $\widetilde{S}$ acts nontrivially on $H_1(\widetilde{S},\mathbb{C})$. We then show that the combination of the lifted actions of $\psi$ and the Galois groups of the covers on $H_1(\widetilde{S},\mathbb{C})$ can be used to determine the Nielsen–Thurston class of $\psi$.

We then turn to the following conjecture: that for each pseudo-Anosov mapping class $\psi$, there exists a lift $\widetilde{\psi}$ to a finite cover whose action on $H_1(\widetilde{S},\mathbb{C})$ has spectral radius strictly greater than one. We show that the conjecture holds if and only if the mapping torus $T_\psi$ has exponential growth of torsion homology with respect to a particular collection of finite covers. We use growth of torsion homology to characterize the mapping classes for which the conjecture holds. Then, we show that if the conjecture fails then $T_\psi$ is large.

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*Dedication.*

*To my high school mathematics teachers: D. Cameron Muir, David Reinstein, Paul J. Sally III and Patricia Wadecki, who always encouraged me to think about mathematics and discover new things.*

# Chapter 1

# Introduction

## 1.1 Overview

Let $S$ be an orientable surface of genus $g$ with $n$ punctures, such that

$$\chi(S) = 2 - 2g - n < 0.$$

We will assume that $S$ has no boundary components, and that the genus and number of punctures of $S$ is finite. The **mapping class group** of $S$, denoted by $\mathrm{Mod}(S)$, is the group of isotopy classes of orientation–preserving homeomorphisms of $S$.

The purpose of this thesis is to study the actions of an individual mapping class $\psi \in \mathrm{Mod}(S)$ on the homology of finite covers of $S$ to which $\psi$ lifts. We are particularly interested in the case where $\psi$ is pseudo-Anosov. The expansion factor of $\psi$ will be written $\lambda(\psi)$. Then, $\log \lambda(\psi) = h(\psi)$ is the minimal topological entropy of a representative of $\psi$. We will use the notation $\widetilde{S}$ for a finite Galois cover of $S$ to which $\psi$ lifts, and $\widetilde{\psi}$ for a lift of $\psi$ to $\widetilde{S}$. The choice of $\widetilde{\psi}$ is well–defined up to pre– and post–composition with an element of the Galois group of $\widetilde{S}$ over $S$.

When considering the action of the lifts of $\psi$ on the homology of covers, we will write

$$\rho(\widetilde{\psi}_*) = \rho(\widetilde{\psi}_* \mid H_1(\widetilde{S}, \mathbb{C})),$$

where the right hand side denotes the spectral radius of the action of $\widetilde{\psi}$ on $H_1(\widetilde{S}, \mathbb{C})$.

## 1.2 Surface conjectures

Our point of departure is the following pair of conjectures, both of which are currently open.

**Conjecture 1.2.1** (Surface conjectures)**.** *Let $\psi \in \mathrm{Mod}(S)$ be pseudo-Anosov. Then:*

- ***(S1)*** *There exists a lift $\widetilde{\psi}$ whose action on $H_1(\widetilde{S}, \mathbb{C})$ has infinite order.*

*In fact:*

- ***(S2)*** *There exists a lift $\widetilde{\psi}$ whose action on $H_1(\widetilde{S}, \mathbb{C})$ satisfies $\rho(\widetilde{\psi}_*) > 1$.*

Clearly, (S2) implies (S1). The surface conjectures can be motivated by the relationship between homology and entropy. If $S$ is a surface and $\psi$ is a pseudo-Anosov homeomorphism with expansion factor $\lambda$, then there is an inequality

$$\rho(\psi_*) \leq \lambda(\psi).$$

The right–hand side of the inequality remains constant under passage to finite covers, but the left–hand side can potentially increase under taking covers. One of the themes in this thesis is to understand how much the left–hand side can increase.

The following result of McMullen ( [33]) shows that the gap in the inequality above may persist over all finite covers. Thus, a strengthening of (S2) which would posit that the supremum of $\rho(\widetilde{\psi}_*)$ over finite covers is equal to $\lambda(\psi)$ is generally false.

**Theorem 1.2.2** (C. McMullen)**.** *For any pseudo-Anosov $\psi$, either:*

1. *(Special case): there is a two–fold lift $\widetilde{\psi}$ such that*

$$\rho(\widetilde{\psi}_*) = \lambda(\psi),$$

   *or*

2. *(General case): we have*

$$\sup \rho(\widetilde{\psi}_*) < \lambda(\psi),$$

   *where the supremum is taken over all lifts of $\psi$ to finite covers $\widetilde{S} \to S$.*

Our first result gives evidence for (S1). It will be proved in Chapter 4 (see also [18]):

**Theorem 1.2.3.** *For each nontrivial mapping class $\psi$, there is a finite cover $\widetilde{S}$ of $S$ such that each lift of $\psi$ to $\mathrm{Mod}(\widetilde{S})$ acts nontrivially on $H_1(\widetilde{S}, \mathbb{C})$.*

## 1.3 Torsion conjectures

To further give evidence for conjectures (S1) and (S2), we relate these statements to well–known conjectures in the theory of 3–manifolds. Given $\psi \in \mathrm{Mod}(S)$, we let

$$T_\psi = S \times I/((s, 0) \sim (\psi(s), 1))$$

be the mapping torus of $\psi$. The fundamental group $\Gamma$ of $T_\psi$ is the semidirect product of $\pi_1(S)$ and $\mathbb{Z}$, with the generator $1 \in \mathbb{Z}$ acting by $\psi$. The isomorphism type of $\Gamma$ is independent of the lift of $\psi$ to $\mathrm{Aut}(\pi_1(S))$ (see for instance [1], Chapter 2, Prop. 12). It is known that $T_\psi$ is hyperbolic (it admits a complete metric of constant negative sectional curvature equal to $-1$) if and only if $\psi$ is pseudo-Anosov.

A **tower** of finite covers of $T$ is a sequence of connected finite covers

$$\cdots \to T_3 \to T_2 \to T_1 \to T.$$

We require $\deg(T_i \to T) \to \infty$ as $i \to \infty$. A sequence of finite index subgroups

$$\cdots \leq \pi_1(T_2) \leq \pi_1(T_1) \leq \pi_1(T)$$

with $[\pi_1(T) : \pi_1(T_i)] \to \infty$ as $i \to \infty$ determines a tower of finite covers of $T$. A tower so determined is called **exhausting** if

$$\bigcap_i \pi_1(T_i) = \{1\}.$$

The following conjectures are suggested by the work of Bergeron–Venkatesh in [3] and of Lück in [26] and [27].

**Conjecture 1.3.1** (Torsion conjectures)**.** *Let $T = T_\psi$ be the mapping torus of a pseudo-Anosov homeomorphism $\psi \in \mathrm{Mod}(S)$. Then:*

- ***(T1)*** *The manifold $T$ has exponential torsion homology growth with respect to some tower of finite covers $\{T_i\}$ of $T$.*

*In fact:*

- ***(T2)*** *The manifold $T$ has exponential torsion homology growth with respect to every tower of exhausting finite covers $\{T_i\}$ of $T$.*

Notice that conjecture (T1) does not require the tower to be exhausting. Conjecture (T2) is Lück's conjecture 1.12 in [27].

A 3–manifold $T$ has **exponential torsion homology growth** with respect to a tower of finite covers $\{T_i\}$ if

$$\limsup_i \frac{\log |H_1(T_i, \mathbb{Z})_{tors}|}{[\deg(T_i \to T)]} > 0.$$

The value of this limit is called the **torsion homology growth exponent**.

We remark that torsion homology growth as described above is special to 3–manifolds. In higher dimensions, homology in dimensions other than one must also be considered. See [3] for a correct definition of torsion homology growth for higher dimensional locally symmetric spaces.

We will show that the torsion conjectures (T1) and (T2) are logically related to the surface conjectures (S1) and (S2) through the following two results:

**Theorem 1.3.2.** *Let $\psi$ be pseudo-Anosov, and suppose $T_\psi$ has exponential torsion homology growth with respect to every exhausting tower of finite covers. Then there is a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$.*

**Theorem 1.3.3.** *Let $\psi$ be pseudo-Anosov, and suppose there exists a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$. Then $T_\psi$ has exponential torsion homology growth with respect to some tower of finite covers.*

Thus for any pseudo-Anosov homeomorphism $\psi$, we have (T2) implies (S2) implies (T1). The proof of Theorem 1.3.2 is short enough to include here:

*Proof of Theorem 1.3.2.* Suppose that no such lift exists. Then for each lift of $\psi$ to a finite cover, we have $\rho(\widetilde{\psi}_*) = 1$. Take any exhausting, nested sequence $\{S_i\}$ of finite covers of $S$ to which $\psi$ lifts. By assumption, $\rho(\widetilde{\psi}_*) = 1$ for each of these covers. For each $i$, there is an exponent $N_i > 0$ such that the action of a lift $\widetilde{\psi}_*^{N_i}$ on $H_1(S_i, \mathbb{C})$ has spectrum $\{1\}$. Since $\rho(\widetilde{\psi}_*^{N_i}) = 1$, we have that if $M_i = i \cdot N_i$ then $\rho(\widetilde{\psi}_*^{M_i}) = 1$ as well. Furthermore, $M_i \to \infty$ as $i \to \infty$. It follows that $H_1(T_{\widetilde{\psi}^{M_i}}, \mathbb{Z})$ is torsion–free. Indeed, any matrix representing $\widetilde{\psi}_*^{M_i}$ is conjugate to an upper triangular matrix with ones down the diagonal. The homology $H_1(T_{\widetilde{\psi}^{M_i}}, \mathbb{Z})$ is given by

$$\mathbb{Z} \oplus H_1(S_i, \mathbb{Z})/(\widetilde{\psi}_*^{M_i} - I),$$

which then has no torsion. Since the $\{S_i\}$ exhaust $S$ and $M_i$ tends to infinity, the covers $\{T_{\widetilde{\psi}^{M_i}}\}$ will exhaust $T_\psi$. $\square$

The proof of Theorem 1.3.2 shows that if (S2) fails then (T2) also fails dramatically: there is an exhausting sequence of finite covers of $T_\psi$ with no torsion homology.

We will discuss Theorem 1.3.2 in more depth and give a proof of Theorem 1.3.3 in Chapter 3. Growth of torsion homology will allow us to characterize mapping classes with the property that each finite lift $\widetilde{\psi}$ satisfies $\rho(\widetilde{\psi}_*) = 1$, intrinsically from the mapping torus. A precise theorem is:

**Theorem 1.3.4.** *Let $T_\psi$ be the mapping torus of a pseudo-Anosov mapping class $\psi$. The following are equivalent:*

1. *The mapping class $\psi$ has a lift $\widetilde{\psi}$ to a finite cover which satisfies $\rho(\widetilde{\psi}_*) > 1$.*
2. *There is a finite cover $X$ of $T_\psi$ such that $X$ has exponential growth of torsion homology with respect to some universal tower of finite abelian covers.*

Here, a **universal tower of abelian covers** of a finite CW complex $X$ is a tower $\{X_i\}$ of finite abelian covers of $X$ with the property that if $X_A \to X$ is any finite abelian cover of $X$, then for all sufficiently large $i$, the cover $X_i \to X$ factors through $X_A \to X$. The **homology covers** of a finite CW complex $X$ are the covers $\{X_N\}$ induced by the canonical maps

$$\pi_1(X) \to H_1(X, \mathbb{Z}/N\mathbb{Z}).$$

A sequence of homology covers $\{X_{N_i}\}$ is a universal tower of abelian covers if and only if $N_i \mid N_{i+1}$ for each $i$, and if for each positive integer $d$, we have that $d \mid N_i$ for all sufficiently large $i$.

We remark that there is a natural notion of an **exhausting tower of abelian covers**, which is related to the notion of a universal tower of abelian covers. An exhausting tower of abelian covers of a finite CW complex $X$ is a tower of covers $\{X_i\}$ determined by a nested sequence of finite index subgroups $\{\Gamma_i\}$ of $\pi_1(X)$ which satisfy

$$\bigcap_i \Gamma_i = [\pi_1(X), \pi_1(X)].$$

In our proof of Theorem 1.3.4, it will not be enough to assume that the tower in part (2) is an exhausting tower of abelian covers – we will need that the tower is a universal tower of abelian covers.

Let $f \in \mathrm{Mod}(S)$ and $g \in \mathrm{Mod}(S')$ be mapping classes, perhaps of different surfaces. We will say that $f$ and $g$ are **commensurable** if the associated mapping tori $T_f$ and $T_g$ are commensurable, which is to say that $T_f$ and $T_g$ are both finitely covered by another 3–manifold $M_{f,g}$. Notice that it follows immediately from the definitions that if $f$ and $g$ are the monodromies associated to different fibrations of a given 3–

manifold, then $f$ and $g$ are commensurable. Notice furthermore that if $f$ and $g$ are commensurable and if $f$ is pseudo-Anosov, then $g$ is automatically pseudo-Anosov.

The following result will be an easy consequence of the proof of Theorem 1.3.4:

**Corollary 1.3.5.** *Let $\psi$ be a pseudo-Anosov mapping class. There exists a lift $\widetilde{\psi}$ to a finite cover satisfying $\rho(\widetilde{\psi}_*) > 1$ if and only if for each mapping class $\phi$ which is commensurable to $\psi$, there exists a lift $\widetilde{\phi}$ to a finite cover satisfying $\rho(\widetilde{\phi}_*) > 1$.*

We will verify Corollary 1.3.5 for an infinite class of examples which arise in [17]. In that paper, Hironaka analyzes the suspension of the mapping class $\psi$ represented by the braid $\sigma_1\sigma_2^{-1} \in B_3$, which is a two–component link complement in $S^3$. She computes the expansion factor of all the monodromies $\{\psi_{(a,b)}\}$ arising from the fibered face which contains the fibration $\psi = \psi_{(1,0)}$. We will show that each such monodromy $\psi_{(a,b)}$ admits a lift $\widetilde{\psi_{(a,b)}}$ satisfying $\rho(\widetilde{\psi_{(a,b)}}) > 1$. We will explore this example in Chapter 3.

## 1.4 3–manifold conjectures

We now explore connections between the surface conjectures (S1) and (S2) and some standard conjectures about 3–manifolds. The results in this section will roughly say that one of the surface conjectures (S1) or (S2) holds for a pseudo-Anosov mapping class $\psi$, or another basic conjecture about 3–manifolds holds for the mapping torus $T_\psi$. Conjectures (M1) and (M2) below are standard, and the reader may consult [36] and [22] for instance.

**Conjecture 1.4.1** (3–manifold conjectures)**.** *Let $T_\psi$ be the mapping torus of a pseudo-*

*Anosov mapping class $\psi$. Then:*

- ***(M1)*** *The manifold $T_\psi$ has virtually infinite Betti number.*

*In fact:*

- ***(M2)*** *The manifold $T_\psi$ is large.*

Here, a manifold $T$ is called **large** if some finite index subgroup of $\pi_1(T)$ maps onto a nonabelian free group. The manifold $T$ has **virtually infinite first Betti number** if for each $N$, there is a finite cover $\widetilde{T}_N$ whose first Betti number is at least $N$. We first give a relatively easy result. Note that it (and indeed all results in this section) can be thought of as "win–win" results, since they say that a surface conjecture holds or a 3–manifold conjecture holds, as opposed to a "lose–win" result:

**Proposition 1.4.2.** *For each mapping class $\psi$, if there exists no lift $\widetilde{\psi}$ with infinite order action on $H_1(\widetilde{S}, \mathbb{C})$ then $T_\psi$ has virtually infinite first Betti number.*

*Proof.* Let $N$ be given. Choose a lift $\widetilde{\psi}$ so that the corresponding cover $\widetilde{S}$ has $\operatorname{rk} H_1(\widetilde{S}, \mathbb{Z}) \geq N$. By assumption, $\widetilde{\psi}$ acts with finite order, so some power $\widetilde{\psi}^k$ of $\widetilde{\psi}$ acts trivially on $H_1(\widetilde{S}, \mathbb{C})$. The suspension of that power of $\widetilde{\psi}$ acting on $\widetilde{S}$ gives the manifold $T_{\widetilde{\psi}^k}$, which covers $T$ and has Betti number at least $N$. □

Thus for each $\psi$, (S1) holds or (M1) holds. Here, the "or" is not exclusive. A deeper result, to be proved in Chapter 5, is the following:

**Theorem 1.4.3.** *Let $\psi$ be pseudo-Anosov, and suppose there exists no lift $\widetilde{\psi}$ to an abelian cover of $S$ with $\rho(\widetilde{\psi}_*) > 1$. Then the manifold $T_\psi$ is large.*

Thus for each pseudo-Anosov $\psi$, (S2) holds or (M2) holds, where again the "or" is not exclusive.

## 1.5 Complementary results on largeness

We will prove the following characterization finitely presented large groups:

**Theorem 1.5.1.** *Let $\Gamma$ be a finitely presented group. The following are equivalent:*

1. *The group $\Gamma$ is large.*
2. *There is a finite index subgroup $\Gamma' < \Gamma$ such that for each $N$, there is a finite index normal subgroup $\Gamma'_N < \Gamma'$ with $\Gamma'/\Gamma'_N$ abelian and satisfying $b_1(\Gamma'_N) \geq N$.*
3. *There exists a finite index subgroup $\Gamma' < \Gamma$ such that the Alexander variety $V_1(\Gamma')$ contains a torsion translate of a rational torus of dimension at least one.*

Here, a **torsion translate of a rational torus** in $V_1$ is a subspace $T \subset V_1 \subset (\mathbb{C}^*)^r$ contained in $(S^1)^r$ which is a smooth subtorus for which torsion points in $(S^1)^r$ are dense in the usual topology.

Theorem 1.5.1 gives the extent to which conjecture (M1) implies (M2) for general finite CW complexes: a finite CW complex is large if and only if it has virtually infinite first Betti number with respect to fixed–finite–by–finite abelian covers. The equivalence of parts (2) and (3) of Theorem 1.5.1 is originally due to Sarnak (see [35], also [14] and [15]). For the convenience of the reader, we will include a self–contained proof of the equivalence of (2) and (3). Once we have established Theorem 1.5.1, we will be able to give an efficient proof of Theorem 1.4.3 by showing that if $\psi$ does not satisfy (S2) then $T_\psi$ satisfies part (2) of Theorem 1.5.1.

## 1.6 The Nielsen–Thurston classification

The following result is an amplification of Theorem 1.2.3. Since for each nontrivial mapping class there exists a finite cover $\widetilde{S}$ such that each lift $\widetilde{\psi}$ acts nontrivially on $H_1(\widetilde{S}, \mathbb{C})$, the homological action of all lifts of $\widetilde{\psi}$ should encode all information about $\psi$. For instance, despite the generally strict inequality between the expansion factor $\lambda(\psi)$ and $\sup \rho(\widetilde{\psi}_*)$ as demonstrated by McMullen's Theorem (Theorem 1.2.2), the expansion factor of a pseudo-Anosov mapping class must be encoded somehow.

Theorem 1.6.1 below shows how to recover the Nielsen–Thurston classification of $\psi$.

**Theorem 1.6.1.** *Let* $\mathrm{Mod}(S)$ *denote a mapping class group of a surface* $S$ *and let* $\psi$ *be a nonidentity mapping class.*

1. *The mapping class* $\psi$ *has finite order if and only if*

$$\sup_{\widetilde{S} \to S} |\widetilde{\psi}| < \infty,$$

   *where* $|\widetilde{\psi}|$ *denotes the order of* $\widetilde{\psi}$ *as an automorphism of the characters of the Galois group representations occurring in* $H_1(\widetilde{S}, \mathbb{C})$.

2. *The mapping class* $\psi$ *is reducible if and only if there is an* $N > 0$ *and a non-peripheral* $g \in \pi_1(S)$ *such that for each characteristic quotient* $G$ *of* $\pi_1(S)$ *and every irreducible character* $\chi$ *of* $G$*, we have*

$$\chi(g) = \chi(\psi^N(g)).$$

3. *The class* $\psi$ *is pseudo-Anosov if both (1) and (2) fail.*

Here, an element of $\pi_1(S)$ is **nonperipheral** if it is not freely homotopic to a puncture of $S$.

## 1.7 Aut **and** Out **for free groups**

There is a parallel and nearly analogous discussion to what we have developed above which applies to automorphisms of free groups. If $F_n$ is a free group and $H < F_n$ is a finite index normal subgroup with quotient $G$, then each irreducible representation of $G$ occurs as a direct summand of the $G$–module $H_1(H, \mathbb{C})$. We will prove an analogue of Theorem 1.6.1 by exploiting this fact. If $H$ is characteristic, one can pose the same questions about the nature of the action of automorphisms on $H_1(H, \mathbb{C})$, as one does for mapping classes. All the results of this thesis concerning those actions carry over verbatim to automorphisms of free groups.

If $\psi \in \mathrm{Out}(F_n)$, one can form the group $G_\psi$ as the semidirect product

$$1 \to F_n \to G_\psi \to \mathbb{Z} \to 1,$$

where the $\mathbb{Z}$–conjugation action is by a lift of $\psi$ to $\mathrm{Aut}(F_n)$. The isomorphism type of $G_\psi$ is independent of the choice of lift. One can certainly ask about the growth of Betti numbers of finite index subgroups of $G_\psi$, the largeness of $G_\psi$ and the growth of torsion homology of finite index subgroups of $G_\psi$. Each result in this thesis which is proved for mapping classes concerning those properties of mapping tori holds verbatim with "mapping class" replaced by "free group automorphism".

We also remark that it would be very interesting to establish an analogue of McMullen's Theorem [33] for free group automorphisms. A plausible statement would be:

**Conjecture 1.7.1.** *Let $\psi \in \mathrm{Out}(F_n)$ be an irreducible automorphism with irreducible powers and expansion factor $\lambda(\psi)$. Either:*

1. *(Special case): there exists a lift $\widetilde{\psi}$ such that*

$$\rho(\widetilde{\psi}_*) = \lambda(\psi),$$

   *or*

2. *(General case): we have*

$$\sup \rho(\widetilde{\psi}_*) < \lambda(\psi),$$

   *where $\widetilde{\psi}$ ranges over all finite lifts of $\psi$.*

McMullen's proof uses the period map from the moduli space of curves to the moduli space of Jacobians. An analogue of the Jacobian for a graph has been developed (see for instance the work of Baker and Norine in [2]).

## 1.8 Notes and references

The expansion factor for a pseudo-Anosov homeomorphism $\psi$ is closely related to its topological entropy. If $\lambda(\psi)$ is the expansion factor then $h(\psi) = \log \lambda(\psi)$ is the minimal topological entropy of a representative in the isotopy class of $\psi$. For homeomorphisms of compact surfaces, the inequality

$$\log \rho(\psi) \leq h(\psi)$$

was established by Manning in [29]. For diffeomorphisms of more general compact manifolds, the same inequality is obtained by Yomdin in [41].

Both conjectures (M1) and (M2) can be stated for general finite–volume hyperbolic orbifolds. Lackenby's survey [22] contains both (M1) and (M2) as conjectures 2 and 5 in the introduction and outlines the current state of affairs, but these conjectures have been known since at least the work of Thurston in [36]. Both conjectures are closely connected with other central problems in 3–manifold theory such as the virtually fibered and virtually Haken conjectures, both of which were stated in [36]. Conjecture (M1) for genus 2 fibered hyperbolic 3–manifolds was resolved by Masters in [30], but is open in general.

The torsion conjectures are more recent, and some recent work on them includes that of Bergeron and Venkatesh in [3] and of Lück in [26] and [27]. Bergeron and Venkatesh establish positive torsion homology growth for congruence covers of arithmetic hyperbolic 3–manifolds, but their ring of coefficients is not $\mathbb{Z}$. They conjecture positive torsion homology growth for more general classes of locally symmetric spaces and more general rings of coefficients. As noted above, conjecture (T2) is Lück's conjecture 1.12 in [27].

In [10], Farb, Leininger and Margalit study **short** dilatation pseudo-Anosov mapping classes, showing that they all come from fibrations of Dehn fillings on finitely many 3–manifolds. Here, short means that one fixes a number $M > 1$ and considers the mapping classes on all surfaces which satisfy $\lambda(\psi)^{|\chi(S)|} \leq M$. In light of Corollary 1.3.5, it is interesting to ask whether the set of short dilatation pseudo-Anosov mapping classes fall into finitely many commensurability classes. The answer is probably no.

The ideas behind the proof of Theorem 1.5.1 fit into a more general discussion of polynomial periodicity. A function $f : \mathbb{N} \to \mathbb{Z}$ is **polynomial periodic** if there is a finite collection $\{p_0, \ldots, p_{n-1}\}$ of polynomials such that if $m \equiv i \pmod{n}$ then

$$f(m) = p_i(m).$$

In [35], Sarnak studied the growth of Betti numbers of congruence covers of finite CW complexes (which we call homology covers in this thesis), which is to say covers $X_N \to X$ induced by the natural map $\pi_1(X) \to H_1(X, \mathbb{Z}/N\mathbb{Z})$. The primary result of his work is that the first Betti numbers of the covers $\{X_N\}$ are polynomial periodic in $N$ (cf. Hironaka in [14]). Laurent's Theorem ( [24]) and Alexander theory allows us to conclude that if $X$ has unbounded growth of $b_1$ as we vary over finite abelian covers, then $X$ has at least linear growth of $b_1$ under finite abelian covers. More careful analysis of various finite abelian covers of $X$ allows us to prove largeness.

There is recent work of D. Wise on quasi–convex hierarchies of hyperbolic groups (see [39], [40], [19]). In particular, his work implies that a fibered 3–manifold with first Betti number at least two is large. The work in this thesis is completely independent of Wise's work.

The results in this thesis which connect the surface conjectures to the torsion and 3–manifold conjectures can be summarized in the following proposition, which roughly says that the homology of finite covers of a fibered hyperbolic 3–manifold $T_\psi$ gets complicated in at least one of two ways – either the first Betti number grows quickly or the torsion part of the homology grows quickly:

**Proposition 1.8.1.** *Let $T_\psi$ be a fibered hyperbolic 3–manifold. Then there exists a* *tower of finite Galois covers*

$$\cdots \to T_2 \to T_1 \to T$$

*such that at least one of the following two conclusions holds:*

1. *We have*
$$\inf_i \frac{b_1(T_i)}{\deg(T_i \to T)} > 0.$$

2. *We have*
$$\limsup_i \frac{\log |H_1(T_i, \mathbb{Z})_{tors}|}{\deg(T_i \to T)} > 0.$$

*Furthermore, we may assume that for each $i$, the cover $T_i \to T_1$ is abelian.*

*Proof.* If there exists a lift $\widetilde{\psi}$ to a finite cover of the fiber such that $\rho(\widetilde{\psi}_*) > 1$ then Theorem 1.3.4 guarantees conclusion (2). If no such lift exists, Theorem 1.4.3 guarantees conclusion (1). □

## 1.9 Outlines of proofs of the main results

To give the reader a feeling for the contents of the rest of the thesis, we will outline some of ideas in the proofs of the main results.

### 1.9.1 Nontriviality of the action of $\mathrm{Mod}(S)$ on homology of finite covers, proofs

The proof of Theorem 1.2.3 proceeds roughly as follows. If $G$ is a finite characteristic quotient of $\pi_1(S)$ then one gets a map $\mathrm{Mod}(S) \to \mathrm{Out}(G)$. The proof the follows from two observations. The first is that the map

$$\mathrm{Mod}(S) \to \prod_{\pi_1(S)\to G} \mathrm{Out}(G)$$

is injective, as $G$ ranges over finite characteristic quotients of $\pi_1(S)$. The second is that if $\widetilde{S} \to S$ is a finite cover with Galois group $G$ then $G$ acts faithfully on $H_1(\widetilde{S}, \mathbb{C})$.

The idea of the proof of Theorem 1.6.1 is also quite simple. If a fundamental group automorphism acts trivially on all characters of all finite characteristic quotients of $\pi_1(S)$ then one can show that it is just conjugation by an element of $\pi_1(S)$. This uses residual finiteness of $\mathrm{Mod}(S)$, conjugacy separability of $\pi_1(S)$, and the fact that irreducible complex characters span the space of class functions in a finite group. If there is a conjugacy class $[g] \subset \pi_1(S)$ which is preserved by $\psi$ then $\psi$ is reducible. Whether or not a given conjugacy class is preserved by a given mapping class can be detected by looking at actions of outer automorphisms on the characters of finite characteristic quotients, since fundamental groups of surfaces are conjugacy separable.

The proofs of all the results discussed in this subsection can be found in Chapter 4.

### 1.9.2 The torsion conjectures, proofs

To establish Theorem 1.3.2, we show through direct computation that if $\rho(\psi) = 1$ then the torsion homology of $T_{\psi^n}$ is $O(1)$ as $n$ tends to infinity. From this observation, we can quickly prove the result. The proof of Theorem 1.3.3 is an easy application of Mahler measure. The proof of Theorem 1.3.4 combines the proofs of Theorems 1.3.2 and 1.3.3, together with some facts about the Alexander polynomial. The proofs appear in Chapter 3.

### 1.9.3 The 3–manifold conjectures, proofs

The proof of Theorem 1.4.3 will follow from the observation that if $\rho(\widetilde{\psi}_*) = 1$ for all finite lifts of $\psi$ then condition (2) of Theorem 1.5.1 will be satisfied. Theorem 1.5.1 is proved by directly analyzing the relationship between Betti numbers of finite covers and torsion solutions to the Alexander polynomial. We will then use work of Lackenby on homology gradients for towers of covers to establish the largeness of the spaces in question. The proofs appear in Chapter 5.

# Chapter 2

# Background

In this Chapter, we will recall some fairly well–known facts which we will use over and over in this thesis. We will generally not supply proofs but references will be given.

## 2.1 Entropy of automorphisms of groups

A good reference for this subsection is [12]. Let $G$ be a group generated by a finite set $S$, and let $\alpha$ be an automorphism of $G$. The set $S$ gives rise to a length function $\ell_S$, which assigns to each $g \in G$ the length of the smallest word in elements of $S$ which represents $g$. Equivalently, $\ell_S(g)$ is the distance between the identity and $g$ in the Cayley graph $\Gamma(G, S)$.

Let $g \in G$ be arbitrary. The sequence $\{\ell_S(\alpha^n(g))\}$ is submultiplicative, so that the limit

$$\lim_{n\to\infty} (\ell_S(\alpha^n(g)))^{1/n}$$

exists and is at least one. Let

$$\lambda_S = \max_{s \in S} (\ell_S(\alpha^n(s)))^{1/n}.$$

We call $\lambda = \lambda_S$ the **growth rate** of $\alpha$. We note that in literature, the **entropy** $h(\alpha)$ is defined as $\log \lambda$. We summarize some basic facts about growth rates of automorphisms for the convenience of the reader:

- The real number $\lambda_S$ is independent of $S$.
- If $G$ is finitely generated and free abelian and if $\alpha \in \mathrm{Aut}(G)$ then $\lambda = \rho(\alpha)$, where the right hand side is the spectral radius of the matrix $\alpha$.
- If $G$ is finitely generated then the value of $\lambda$ depends only on the image of $\alpha$ in $\mathrm{Out}(G)$. In particular, if $\alpha$ is inner then $\lambda = 1$.
- Growth rates are non–increasing under taking quotients. Precisely, let $G$ be a finitely generated group and $Q$ be a quotient of $G$. Suppose $\alpha \in \mathrm{Aut}(G)$ and that $\alpha$ descends to an automorphism of $Q$. Then the growth rate of $\alpha$ as an automorphism of $G$ is at least as large as the growth rate of $\alpha$ as an automorphism of $Q$.
- (Compare with Subsection 2.3); if $\psi \in \mathrm{Aut}(\pi_1(S))$ descends to a pseudo-Anosov mapping class of $S$, then the growth rate of $\psi$ coincides with the expansion factor of the corresponding mapping class. Furthermore, for each $g \in \pi_1(S)$ we have $\ell(\psi^n(g)) \sim \lambda(\psi)^n$.

A good reference for the previous facts is [12], for instance.

## 2.2 Eigenvalues of integer matrices

We will be appealing repeatedly to the following fact from number theory which is originally due to Kronecker:

**Proposition 2.2.1.** *Let $M \in GL_n(\mathbb{Z})$ and let $\lambda$ be an eigenvalue of $M$ of length one. Suppose that $\lambda$ is not a root of unity. Then the spectral radius of $M$ is greater than one.*

*Proof.* If $\lambda$ is any eigenvalue of $M$ then its minimal polynomial must divide the characteristic polynomial of $M$. In particular, all the Galois conjugates of $\lambda$ are also eigenvalues of $M$. It is well–known that if $\lambda$ is an algebraic integer of length one, all of whose Galois conjugates also have length one, then $\lambda$ is a root of unity.

Indeed, let $\{\alpha_1, \cdots, \alpha_k\}$ be the roots of the minimal polynomial of $\lambda$, each of which has norm one. For each $n$, we define

$$p_n(z) = \prod_{i=1}^{k} (z - \alpha_i^n).$$

The coefficients of $p_n(z)$ are symmetric functions of $\{\alpha_1, \cdots, \alpha_k\}$ and are therefore integral, and the coefficients are bounded since each $\alpha_i$ has length one. It follows that the collection $\{p_n(z)\}$ is a finite collection of polynomials. We have $p_m(z) = p_n(z)$ for some $n \neq m$. Without loss of generality, $n$ divides $m$. In particular, raising each $\alpha_i^n$ to an integral power permutes the list $\{\alpha_1^n, \ldots, \alpha_k^n\}$. Replacing $m$ by a larger exponent if necessary shows that for each $i$, we have $\alpha_i^n = \alpha_i^m$. In particular, $\alpha_i$ is a root of unity. $\square$

## 2.3 The classification of mapping classes

Let $\psi \in \mathrm{Mod}(S)$. Then $\psi$ has either finite order or infinite order in $\mathrm{Mod}(S)$. In the latter case, $\psi$ is called **finite order**.

**Proposition 2.3.1.** *Let $\psi$ be a finite order mapping class, and suppose $S$ admits at least one complete hyperbolic metric of finite volume. Then there exists a hyperbolic metric on $S$ and a representative $\tilde{\psi}$ in the homotopy class of $\psi$ which is an isometry in that metric and hence has finite order as a homeomorphism of $S$.*

Henceforth in this section, assume that $\psi$ has infinite order. We have that $\psi$ acts on the set of homotopy classes of simple, essential, nonperipheral (puncture–parallel) closed curves on $S$. If $\psi$ has a finite orbit in this set, we say that $\psi$ is **reducible**. If $\psi$ has no such finite orbits, we say that $\psi$ is pseudo-Anosov. A mapping class is called **pure** if there is a (possibly empty) multicurve $\mathcal{C}$ such that $\psi$ stabilizes each component of $S \setminus \mathcal{C}$ and each component of $\mathcal{C}$, and such that the restriction of $\psi$ to any component of $S \setminus \mathcal{C}$ is either the identity or is a pseudo-Anosov homeomorphism. Each mapping class has a power which is pure.

Pseudo-Anosov homeomorphisms have many equivalent definitions. For instance, they stabilize a pair of transverse measured foliations $\mathcal{F}^{\pm}$ which they stretch and contract respectively by a real algebraic integer $\lambda > 1$, called the **expansion factor** of $\psi$.

**Proposition 2.3.2.** *The expansion factor of a pseudo-Anosov homeomorphism $\psi$ coincides with its growth rate as an outer automorphism of $\pi_1(S)$.*

An interesting feature of pseudo-Anosov homeomorphisms is that if we take any nontrivial, nonperipheral $\gamma \in \pi_1(S)$, then

$$\lim_{n\to\infty} (\ell(\psi^n(\gamma)))^{1/n} = \lambda,$$

independently of the generating set for $\pi_1(S)$ and the choice of $\gamma$.

We can summarize the classification as follows:

**Theorem 2.3.3.** *Let $\psi \in \mathrm{Mod}(S)$.*

1. *The mapping class $\psi$ is finite order if and only if some isotopy representative of $\psi$ has finite order as an isometry of some complete, finite volume hyperbolic metric on $S$.*
2. *The mapping class $\psi$ is reducible if and only if there is a multicurve $\mathcal{C} \subset S$ such that, up to isotopy, $\psi(\mathcal{C}) = \mathcal{C}$.*
3. *The mapping class $\psi$ is pseudo-Anosov if and only if it is neither finite order nor reducible.*

Good references for the proof of this result are [6], [12], [11] and [38].

## 2.4 McMullen's spectral gap theorem

The reference for this section is [33]. Throughout this section, let $\psi$ be a pseudo-Anosov homeomorphism. In the previous chapter, we mentioned a distinction between **homological** and **non–homological** pseudo-Anosov homeomorphisms. The former class is characterized by orientable stable and unstable foliations $\mathcal{F}^{\pm}$, and equivalently by the invariant quadratic differential $q$ being globally the square of a one–form $\omega$. The spectral radius of the action of $\psi$ on $H_1(S, \mathbb{R})$ is equal to $\lambda$.

It is occasionally true that $\psi$ is non–homological but becomes homological after passing to a finite cover of $S$. This happens if and only if the foliations $\mathcal{F}^{\pm}$ have only even–order singularities except for possibly one–pronged singularities at the punctures.

If $\psi$ is a non–homological pseudo-Anosov then the spectral radius of the action of $\psi$ on $H_1(S, \mathbb{R})$ is strictly smaller than the expansion factor (see [20] and [23]).

McMullen proved the following result:

**Theorem 2.4.1.** *Suppose $\psi$ is non–homological on each finite cover of $S$. Then the supremum over lifts to finite covers*

$$\sup \rho(\widetilde{\psi}_*)$$

*is strictly less than the expansion factor $\lambda(\psi)$.*

## 2.5 Residual finiteness, conjugacy separability and Galois group actions on finite covers

The fundamental group $\pi_1(S)$ is **residually finite**, meaning that each nonidentity element of $\pi_1(S)$ persists in some finite quotient of $\pi_1(S)$. It follows that $S$ has many finite **characteristic** covers, which is to say ones which are invariant under all automorphisms of $\pi_1(S)$. Let $G$ be a finite characteristic quotient of $\pi_1(S)$. Because of the identification of $\mathrm{Mod}(S)$ with a subgroup of $\mathrm{Out}(\pi_1(S))$, we obtain a map

$$\mathrm{Mod}(S) \to \mathrm{Out}(G).$$

There is an action of $\mathrm{Out}(G)$ on the set of irreducible characters $X(G)$ of $G$. The composition map

$$\mathrm{Mod}(S) \to \mathrm{Out}(G) \to \mathrm{Sym}(X(G))$$

is "eventually faithful", in the sense that each nontrivial element of $\mathrm{Mod}(S)$ acts nontrivially on $X(G)$ for some such quotient $G$. Furthermore, each irreducible representation of $G$ occurs as a summand of $H_1(\widetilde{S}, \mathbb{C})$, where $\widetilde{S}$ is the cover corresponding to $G$. This is roughly the reason for which each mapping class acts nontrivially on the homology of some finite cover of $S$.

We will write $\tau = \tau(G)$ for the map $\mathrm{Mod}(S) \to \mathrm{Sym}(X(G))$ for a finite characteristic quotient $G$ of $\pi_1(S)$.

We remark that in general it is not clear that $\mathrm{Out}(G)$ acts faithfully on $X(G)$. It is possible to find a group $G$ and a non–inner automorphism $f$ for which $f(g)$ is conjugate to $g$ for each $g \in G$. Since characters are class functions, such an automorphism acts trivially on $X(G)$.

In [13], Grossman proves that $\mathrm{Out}(F_n)$ and $\mathrm{Mod}(S)$ are residually finite groups. To do this, she establishes that if $\psi \in \mathrm{Aut}(\pi_1(S))$ and preserves the conjugacy class of each element of $\pi_1(S)$ then it is an inner automorphism. A similar result is proven for automorphisms of free groups. Thus, for each mapping class $\psi$ there is a conjugacy class $[g] \subset \pi_1(S)$ which is not preserved by $\psi$.

Thus to prove that

$$\prod_G \tau(G) : \mathrm{Mod}(S) \to \prod \mathrm{Sym}(X(G))$$

is injective, as $G$ varies over finite characteristic quotients of $\pi_1(S)$, it suffices to show that if $g, h \in \pi_1(S)$ are not conjugate then there is a finite quotient $G$ where the images of $g$ and $h$ are not conjugate, i.e. **conjugacy separability** for surface groups. The reader may consult [34] or [28], page 26, for a proof of conjugacy separability for surface groups.

## 2.6 Fibered 3–manifolds

Let $M$ be a 3–manifold which fibers of the circle. This means that there is a fibration

$$S \to M \to S^1$$

with the monodromy given by a homeomorphism $\psi$ of $S$. It is well–known and easy to show that the homotopy type of $M = M_\psi$ depends only on the mapping class of $\psi$.

A result of Thurston (see [31] for a lengthy discussion) says that the manifold $M$ admits a hyperbolic metric of finite volume if and only if the monodromy $\psi$ is a pseudo-Anosov homeomorphism. If $\psi$ has finite order then a finite cover of $M$ is homeomorphic to $S \times S^1$, and if $\psi$ is reducible then $M$ contains at least one non–peripheral essential torus. In either of those cases, $M$ is not hyperbolic.

Given $\psi$, one can easily write a presentation for the fundamental group $\pi_1(M)$. The fact that $M$ fits into a fibration implies that the fundamental group of $M$ fits into a short exact sequence

$$1 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 1.$$

We can lift $\psi$ to an automorphism of $\pi_1(S)$ which we also call $\psi$. A presentation can be given as follows:

$$\pi_1(M) = \langle \pi_1(S), t \mid t\pi_1(S)t^{-1} = \psi(\pi_1(S)) \rangle.$$

Similarly, it is easy to compute $H_1(M, \mathbb{Z})$. Clearly, the homology group is a quotient of $\mathbb{Z} \oplus H_1(S, \mathbb{Z})$. On the other hand, each commutator is in the kernel of the abelianization map. So for each $d \in H_1(S, \mathbb{Z})$, the element $\psi(d) - d$ is in the kernel of the abelianization. If $C(\psi)$ denotes the span of these elements in $H_1(S, \mathbb{Z})$, we have

$$H_1(M, \mathbb{Z}) \cong \mathbb{Z} \oplus H_1(S, \mathbb{Z})/C(\psi).$$

For the remainder of this section, [37] is the canonical reference. Let $M$ be a compact, irreducible, atoroidal 3–manifold whose boundary (if any) is a finite union of tori. For any compact surface,

$$S = \bigcup_{i=1}^{n} S_i,$$

write $\chi_-(S)$ for the sum of the absolute values of the Euler characteristics of the components of $S$ which have negative Euler characteristic. If $\phi \in H^1(M, \mathbb{Z})$ is a cohomology class, write

$$||\phi||_T = \min \chi_-(S),$$

where $[S] \in H_2(M, \partial M, \mathbb{Z})$ is dual to $\phi$. This integer is called the **Thurston norm** of $\phi$.

**Theorem 2.6.1.** *The Thurston norm has the following properties:*

1. *The Thurston norm extends to a continuous norm on $H^1(M, \mathbb{R})$.*

2. *The unit ball in the Thurston norm is a finite–sided polyhedron whose vertices are rational.*

Suppose $M$ fibers over the circle. Then the fibration $M \to S^1$ yields a homomorphism $\pi_1(M) \to \mathbb{Z}$, which is a cohomology class. Such a cohomology class is called **fibered**. The homology class of the fiber is dual to this cohomology class and is norm–minimizing.

**Theorem 2.6.2.** *The following are properties of a fibered cohomology class $\phi$:*

1. *The cohomology class $\phi$ is an integral point in the interior of a cone $C$ over a top–dimensional face of the unit norm ball.*

2. *Every primitive integral cohomology class in the interior of $C$ is also a fibered cohomology class.*

The Thurston norm is only interesting from the point of view of inequivalent fibrations if the rank of $H^1(M, \mathbb{R})$ is at least two, for otherwise there is only one primitive cohomology class in the cone over a top–dimensional face.

## 2.7 Mahler measure

The notion of Mahler measure will be useful in studying the growth of torsion homology. Let $p(t) \in \mathbb{Z}[t^{\pm 1}]$ be a monic Laurent polynomial. The **Mahler measure** of $p$ is written

$$M(p) = \prod_i \max\{1, |\alpha_i|\},$$

where $\{\alpha_i\}$ ranges over the roots of $p$. A general fact about Mahler measures is the following:

**Lemma 2.7.1.** *Let $A$ be a square matrix with characteristic polynomial $p_A$, and for any square matrix $M$ let $\det'(M)$ denote the product of the nonzero eigenvalues of $M$. Then*

$$\lim_{n\to\infty} \frac{\log|\det'(A^n - I)|}{n} = \log M(p_A).$$

The relationship between Mahler measure and the size of $H_1(M, \mathbb{Z})_{tor}$ is thus given by the following:

**Lemma 2.7.2.** *Let $A \in GL_n(\mathbb{Z})$ and let*

$$1 \to \mathbb{Z}^n \to \Gamma \to \mathbb{Z} \to 1$$

*be a semidirect product of $\mathbb{Z}$ and $\mathbb{Z}^n$ whose $\mathbb{Z}$–action is given by $A$. Then*

$$|H_1(\Gamma, \mathbb{Z})_{tor}| = |\det{}'(A - I)|.$$

We will need the following elementary linear algebra result about the relationship between torsion homology growth, $\det'$ and matrices:

**Lemma 2.7.3.** *Let $A \in GL_n(\mathbb{C})$ have all eigenvalues on the unit circle. We have the inequality*

$$\log|\det{}'(A - I)| \le n \cdot \log 2.$$

*Proof.* Viewing $A$ as an automorphism of $\mathbb{C}^n$, we may conjugate $A$ to express it as an upper triangular matrix. Each eigenvalue of $\mu$ of $A$ satisfies $|\mu| = 1$, so that $|\mu - 1| \le 2$. The conclusion of the lemma is immediate. □

Note that if $A$ has all of its eigenvalues on the unit circle then so does $A^N$ for each $N$. Notice furthermore that the bound given in Lemma 2.7.3 is sharp and is realized by $-I$, which in the world of mapping classes is realized by the hyperelliptic involution.

## 2.8 Alexander stratifications and Alexander polynomials

Let $G$ be a finitely presented group. We will write

$$G = \langle F_r \mid \mathcal{R} \rangle,$$

where $F_r$ is a free group of rank $r$ and $\mathcal{R}$ is a finite collection of relations. Write $\widehat{G}$ for the group of characters of $G$, i.e. $\mathrm{Hom}(G, S^1)$. The group $\widehat{G}$ has the structure of an algebraic group whose coordinate ring is $\mathbb{C}[G^{ab}]$. Given a map $\alpha : G' \to G$, we get a map

$$\widehat{\alpha} : \widehat{G} \to \widehat{G'}$$

by precomposition, and an induced map

$$\widehat{\alpha}^* : \mathbb{C}[(G')^{ab}] \to \mathbb{C}[G^{ab}].$$

If $g \in G$, write $\overline{g}$ for its image in $G^{ab}$, and write $\Lambda_r(\mathbb{Z})$ for the ring of integral Laurent polynomial rings in $r$ variables over $\mathbb{Z}$. The Fox derivative furnishes maps

$$D_i : F_r \to \Lambda_r(\mathbb{Z})$$

which are defined by

$$D_i(x_j) = \delta_{i,j}$$

and by

$$D_i(fg) = D_i(f) + \overline{f} D_i(g).$$

The $r$–tuple $(D_1, \ldots, D_r)$ is written $D$ and called the **Fox derivative**.

Write $q$ for the natural surjection $F_r \to G$. The **Alexander matrix** of the presentation

$$G = \langle F_r \mid \mathcal{R} \rangle$$

is the matrix of partial derivatives

$$M = M(F_r, \mathcal{R}) = [(\widehat{q})^* D_i(R_j)],$$

where $R_j$ ranges over $\mathcal{R}$.

An alternative viewpoint on the Alexander matrix is as follows. Let $X$ be a finite CW complex whose fundamental group $G$ with abelianization $\Gamma$, and let $Y$ be the corresponding cover of $X$. We have a natural $\Gamma$–action on $Y$. Choose a finite CW structure on $X$. If $X$ has $s$ cells in dimension one and $r$ cells in dimension two, then there is a $\Gamma$–equivariant identification

$$C_1(Y, \mathbb{C}) \cong \mathbb{C}[\Gamma]^s$$

and

$$C_2(Y, \mathbb{C}) \cong \mathbb{C}[\Gamma]^r.$$

The boundary map

$$R : \mathbb{C}[\Gamma]^r \to \mathbb{C}[\Gamma]^s$$

is represented by the Alexander matrix $M$, which can be computed from any presentation of $G$ via the Fox calculus.

If $G \to H$ is a surjection to another abelian group then one similarly obtains a boundary map

$$R_H : \mathbb{C}[H]^r \to \mathbb{C}[H]^s.$$

The map $R$ is related to the map $R_H$ by the universal property of of the Alexander matrix:

**Theorem 2.8.1** (See [14], Section 2)**.** *Let $G$ be a finitely presented group and let*

$$\phi : G \to A$$

*be a surjective map to an abelian group $A$. Then the matrix $\phi(M)$ represents the map $R_A$.*

We write $V_i(G)$ for the characters of $G$ such that the matrix $M(F_r, \mathcal{R})$ has rank less than $r - i$. One obtains a stratification of $\widehat{G}$ by algebraic subsets, called the **Alexander stratification**:

$$\widehat{G} \supset V_1(G) \supset \cdots \supset V_r(G).$$

If the abelianization of $G$ is torsion–free, one can define the **Alexander polynomial** $A(G)$ as the greatest common divisor of the $(r-1) \times (r-1)$ minors of the Alexander matrix over the ring $\mathbb{Z}[G^{ab}]$. The Alexander polynomial defines the largest hypersurface contained in $V_1$. If $G$ has torsion in its abelianization, one can project to the largest torsion–free quotient of $G$ and use the induced Alexander matrix guaranteed by Theorem 2.8.1. This will be the general definition of the Alexander polynomial.

By Theorem 2.8.1, any surjection $\phi : G \to A$ to an abelian group which factors through the universal torsion–free abelian quotient of $G$ gives rise to a specialization of the Alexander polynomial under $\phi$, given by $\phi(A(G)) \in \mathbb{Z}[A]$. Note that this definition makes sense even if $A$ has torsion.

The Alexander stratification is useful for computing the first Betti number of finite abelian covers of finite CW complexes:

**Theorem 2.8.2** ( [15], Proposition 2.5.6)**.** *Let $X$ be a finite CW complex such that $G = \pi_1(X)$, let $\alpha : G \to \Gamma$ be a map onto a finite abelian group, and let $X_\alpha$ be the finite cover of $X$ induced by $\alpha$. Then*

$$b_1(X_\alpha) = b_1(X) + \sum_{i=1}^{r} |V_i(G) \cap \widehat{\alpha}(\widehat{\Gamma} \setminus \widehat{1})|.$$

Thus the first Betti numbers of finite abelian covers of a finite CW complex can be computed from the torsion points in the Alexander strata. Since the vanishing locus of the Alexander polynomial of $X$ is contained in $V_1$, any torsion points which are roots of the Alexander polynomial contribute to the first Betti number of finite covers of $X$.

For any algebraic subset $V$ of an affine torus, write $\mathrm{Tor}(V)$ for the torsion points contained in $V$. The following result of Laurent ( [24]) is helpful for understanding the torsion points of an algebraic subset of an affine torus $(\mathbb{C}^*)^r$:

**Theorem 2.8.3** ( [15], Theorem 4.1.1)**.** *If $V \subset (\mathbb{C}^*)^r$ is any algebraic subset, then there exist rational planes $P_1, \ldots, P_k$ in $(\mathbb{C}^*)^r$ such that each $P_i \subset V$ and such that*

$$\mathrm{Tor}(V) = \bigcup_{i=1}^{k} \mathrm{Tor}(P_i).$$

If $\widetilde{S}$ is a finite cover of a surface $S$ with abelian deck group $\Gamma$ then $H_1(\widetilde{S}, \mathbb{C})$ splits into eigenspaces corresponding to the irreducible characters of the deck group. For an irreducible character $\chi$, the $\chi$–eigenspace can be identified with the twisted homology group $H_1(S, \mathbb{C}_\chi)$.

Suppose that $b_1(T_\psi) \geq 2$. We write

$$H = \mathrm{Hom}(H^1(S, \mathbb{Z})^\psi, \mathbb{Z}) \neq 0,$$

where $H^1(S,\mathbb{Z})^\psi$ is the $\psi$–invariant cohomology of $S$. We will be interested in finite covering spaces which arise from finite quotients of $H$. If $\chi$ is a finite character of $H$, it turns out that the action of $\psi$ on $H_1(S,\mathbb{C}_\chi)$ is governed by the Alexander polynomial.

The group $G = H_1(T_\psi,\mathbb{Z})/torsion$ decomposes as a direct sum $H \oplus \mathbb{Z}$, where the $\mathbb{Z}$–summand is dual to monodromy class $[\psi]$. We will write $t$ for a generator of this $\mathbb{Z}$. The Alexander polynomial $A$ of $T_\psi$ with respect to the quotient $G$ of $\pi_1(T_\psi)$ is an element of the Laurent polynomial ring $\mathbb{Z}[G]$.

**Theorem 2.8.4** ( [32], Corollary 3.2 and discussion immediately following)**.** *Let $\chi$ be a character of $H$. Then the characteristic polynomial of the action of $\psi$ on $H_1(S,\mathbb{C}_\chi)$ is given by $\chi(A)$.*

The meaning of $\chi(A)$ is that we substitute $\chi(h)$ for every element of $H$ occurring in $A$ and leave $t$ untouched.

## 2.9 Dependence on the lift

Let $\psi \in \mathrm{Mod}(S)$ and let $\widetilde{S}$ be a finite Galois cover of $S$ with Galois group $G$, and suppose $\psi$ lifts to a mapping class $\widetilde{\psi}$ of $\widetilde{S}$. There is inherent ambiguity in the choice of the lift. Indeed, $\psi$ is only an outer automorphism of $\pi_1(S)$ and of $G$. The fundamental group $\pi(S)$ acts on $\widetilde{S}$ via the Galois group $G$. The quotient of $\widetilde{S}$ by $G$ is $S$. Thus, if $\widetilde{\psi}$ is pre–composed or post–composed with an element of $G$, the resulting homeomorphism of $\widetilde{S}$ is still a lift of $\psi$. We can write down all lifts of $\psi$ by choosing an arbitrary lift $\widetilde{\psi}$ and then considering the elements

$$\{g \cdot \widetilde{\psi} \cdot h\},$$

where $g, h \in G$.

We thus obtain a natural map

$$\mathrm{Mod}(S) \to G \cdot \mathrm{Mod}(\widetilde{S}) \cdot G$$

which sends a mapping class of $S$ to its set of lifts. The set of lifts of a mapping class $\psi$ to $\mathrm{Mod}(\widetilde{S})$ is therefore the double coset $G\widetilde{\psi}G$.

**Proposition 2.9.1.** *Let $\widetilde{S}$ be a finite, Galois cover of $S$ to which $\psi$ lifts, as above.*

1. *If some lift of $\psi$ acts with infinite order on $H_1(\widetilde{S}, \mathbb{C})$, then all lifts do.*

2. *If some lift of $\psi$ acts with spectral radius greater than one on $H_1(\widetilde{S}, \mathbb{C})$, then all lifts do. The spectral radii of all the lifts are the same.*

3. *The Nielsen–Thurston classification of $\psi$ is the same as the classification of any of its lifts. The expansion factor of a lift of a pseudo-Anosov $\psi$ agrees with the expansion factor on the base surface.*

*Proof.* The naturality of the map which associates a mapping class its collection of lifts implies that a power of a lift of a mapping class is a lift of a power of a mapping class. Thus, if we prove the proposition for some nonzero power of $\psi$, we have proved it for $\psi$. Let $\widetilde{\psi}$ be an arbitrary lift of $\psi$. Now consider a lift $g\widetilde{\psi}h$ of $\psi$ and the square

$$(g\widetilde{\psi}h)^2 = g\widetilde{\psi}hg\widetilde{\psi}h.$$

We can slide the first $h$ to the left, and the $\psi$–action permutes the cosets to give another double coset $gh'\widetilde{\psi}g\widetilde{\psi}h$. Similarly, we can slide the second $g$ to the right to get $gh'\widetilde{\psi}^2g'h$. Since the Galois group is finite, some power of $\widetilde{\psi}$ induces the trivial permutation of the cosets. By the naturality of the map which associates to a mapping class its lifts, we can assume that we started with a power of $\psi$ which already trivially permuted the cosets, so that

$$(g\widetilde{\psi}h)^2 = gh\widetilde{\psi}^2 gh.$$

If the element $gh$ has order $n$ in the Galois group, then

$$(gh\widetilde{\psi}^2 gh)^n = (gh)^n \widetilde{\psi}^{2n}(gh)^n = \widetilde{\psi}^{2n}.$$

Thus, the action of $(g\widetilde{\psi}h)^{2n}$ coincides with that of $\widetilde{\psi}^{2n}$. The choice of $g$ and $h$ was arbitrary. Therefore, if one lift of $\psi$ has infinite order as an automorphism of $H_1(\widetilde{S}, \mathbb{C})$ then each lift does. Similarly, if one lift has spectral radius $\lambda$ then all lifts do.

Suppose that $\psi$ has finite order, with order $N$. Then the lifts of $\psi^N$ are the lifts of the identity, which are just the double coset $G \cdot 1 \cdot G$. Since $G$ is finite, any element of the double coset has finite order. Suppose that $\psi$ is reducible. Then $\psi$ stabilizes a multicurve $\mathcal{C}$. If $\widetilde{\mathcal{C}}$ is the preimage of $\mathcal{C}$ in $\widetilde{S}$, observe that $\widetilde{\mathcal{C}}$ is a multicurve and that $\widetilde{\mathcal{C}}$ is $\widetilde{\psi}$–invariant for any lift of $\psi$. Finally suppose $\psi$ is pseudo-Anosov with invariant foliations $\mathcal{F}^{\pm}$. Then each lift $\widetilde{\psi}$ stabilizes a pair of lifted foliations $\widetilde{\mathcal{F}}^{\pm}$, which are locally stretched and contracted respectively by the expansion factor $\lambda$. Each leaf of $\widetilde{\mathcal{F}}^{\pm}$ non–compact, since a closed leaf would descend to a closed leaf on $S$ preserved by $\psi$. It follows that $\widetilde{\psi}$ is pseudo-Anosov. $\square$

Note that the proof of the previous proposition requires passing to powers in general. Thus, the nontriviality of the action of each lift of $\psi$ is not immediate from the nontriviality of the action of $\psi$, and this is why we need to do more work to show that each lift of a nontrivial mapping class acts nontrivially on the homology of some finite cover.

# Chapter 3

# Growth of torsion homology

In this chapter, we prove the results relating the surface conjectures and the torsion conjectures:

**Theorem 3.0.2** (Theorem 1.3.2)**.** *Let $\psi$ be pseudo-Anosov and suppose that the mapping torus $T_\psi$ has exponential torsion homology growth with respect to every exhausting tower of finite covers. Then there exists a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$.*

**Theorem 3.0.3** (Theorem 1.3.3)**.** *Let $\psi$ be pseudo-Anosov and suppose that there exists a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$. Then there exists a tower of finite covers of the mapping torus $T_\psi$ with exponential torsion homology growth.*

## 3.1 Cyclotomic fibrations

Let

$$p : T_\psi \to S^1$$

be the fibration with fiber $S$ and monodromy $\psi$. Suppose that for each lift $\widetilde{\psi}$ to a finite cover $\widetilde{S}$, we have $\rho(\widetilde{\psi}) = 1$. We call the pair $(T_\psi, p)$ a **cyclotomic fibration**.

We would like to have a somewhat deeper understanding of cyclotomic fibrations, and we shall develop some of the theory here.

Suppose $T_\psi$ is any mapping torus with fiber $S$. There is a natural map

$$\iota : \pi_1(S) \longrightarrow \pi_1(T_\psi)$$

given by the inclusion of the fiber. Standard covering space theory implies the following characterization of the number of components of the preimage of $S$:

**Lemma 3.1.1.** *Let $X \to T_\psi$ be a finite Galois cover given by a map $\phi : \pi_1(T_\psi) \to G$ for some finite group $G$. The number of components of the preimage of $S$ in $X$ is equal to the index of the image of $\phi \circ \iota$ in $G$.*

Now one can give the following estimate about the growth of torsion homology for towers of covers of a cyclotomic fibration:

**Lemma 3.1.2.** *Let $T_i$ be a degree $d_i$ cover of $T_\psi$, and suppose that $(T_\psi, p)$ is a cyclotomic fibration. Then for $d_i$ sufficiently large, we have*

$$\frac{\log |H_1(T_i, \mathbb{Z})_{tors}|}{d_i} \leq \frac{C}{c_i} + O\left(\frac{1}{d_i}\right),$$

*where $C$ is a fixed constant and $c_i$ is the number of components of the preimage of $S$ in $T_i$.*

*Proof.* Suppose that the fiber $S$ of the fibration $p$ is closed of genus $g > 1$. Then the absolute value Euler characteristic of $S$ is $2g-2$. If $\widetilde{S}$ is a component of the preimage of $S$ in $M_i$ then $\widetilde{S}$ covers $S$ with some degree $q_i$. The absolute value of the Euler characteristic of $\widetilde{S}$ is $q_i(2g-2)$, so that its genus is $q_i(g-1)+1$. By Lemma 3.1.1, we have that $q_i$ divides $d_i$ with quotient $c_i$, and $c_i$ is the number of components of the preimage of $S$. The rank of the homology of $\widetilde{S}$ is $2q_i(g-1)+2$. By Lemma 2.7.3, we have that

$$\frac{\log |H_1(M_i,\mathbb{Z})_{tors}|}{d_i} \leq \frac{(\log 2)(2q_i(g-1)+2)}{q_i \cdot c_i} \leq \log 2\frac{2(g-1)}{c_i} + \frac{2\log 2}{d_i}.$$

The argument for when $S$ is not closed is analogous. □

Lemma 3.1.2 implies that if $T_\psi$ admits a cyclotomic fibration then the torsion homology growth rate for any tower of covers of $T_\psi$ is $O(1)$, and that it is $o(1)$ if the number of components of the preimage of the fiber of the cyclotomic fibration tends to infinity.

## 3.2 Mapping tori admitting non–cyclotomic fibrations

Now suppose there is a lift $\widetilde{\psi}$ with $\rho(\widetilde{\psi}_*) > 1$. Let $\widetilde{S}$ be the corresponding finite cover of $S$. There is a finite–by–infinite cyclic cover of $T_\psi$ corresponding to the subgroup $\pi_1(\widetilde{S})$ of $\pi_1(T_\psi)$, and it is homeomorphic to $\widetilde{S} \times \mathbb{R}$. We have that $T_{\widetilde{\psi}}$ is a finite cover of $T_\psi$, and we write $T_k$ for the finite cover $T_{\widetilde{\psi}^k}$ of $T_{\widetilde{\psi}}$.

**Proposition 3.2.1.** *Suppose that $\psi$ admits a lift to a finite cover with $\rho(\widetilde{\psi}_*) > 1$. Then the tower of covers $\{T_k\}$ has exponential torsion homology growth.*

*Proof.* This follows from the fact that $\widetilde{\psi}_*$ has an eigenvalue of the unit circle. It follows that the torsion homology growth rate is given by the logarithm of the Mahler measure of $\widetilde{\psi}_*$, which is positive. $\square$

We immediately obtain the following:

**Corollary 3.2.2** (Theorem 1.3.3)**.** *Suppose that $T_\psi$ admits a non–cyclotomic fibration. Then there exists a tower of finite covers of $T_\psi$ with exponential torsion homology growth.*

## 3.3 A characterization of mapping tori admitting cyclotomic fibrations

In this section we will show that if a mapping torus $T_\psi$ admits one cyclotomic fibration then all fibrations of all finite covers of $T_\psi$ are automatically cyclotomic as well. Let $X$ be a finite CW complex. Recall that a **universal tower of abelian covers** of $X$ is a tower $\{X_i\}$ of finite abelian covers of $X$ with the property that if $X_A \to X$ is any finite abelian cover, the cover $X_i \to X$ factors through $X_A$ for all sufficiently large $i$.

**Theorem 3.3.1** (cf. Theorem 1.3.4)**.** *Let $T_\psi$ be the mapping torus of a mapping class $\psi$. The following are equivalent:*

1. *The bundle $T_\psi$ admits a fibration which is non–cyclotomic.*

2. *Each fibration of $T_\psi$ is non–cyclotomic.*

3. *There exists a finite cover $X$ of $T_\psi$ such that some universal tower of abelian covers of $X$ has exponential torsion homology growth.*

*Proof.* Clearly (2) implies (1). Suppose that $T_\psi$ admits a cyclotomic fibration and that $X$ is any finite cover of $T_\psi$. Fix a fiber $S$ and a cyclotomic fibration monodromy $\widetilde{\psi}$ of $X$. In any universal tower of abelian covers of $X$, the number of components in the preimage of $S$ tends to infinity. Indeed, in the homology cover $X_N$ induced by $\pi_1(X) \to H_1(X, \mathbb{Z}/N\mathbb{Z})$, the number of preimages of the fiber is at least $N$. By definition, some level $X_i$ of the universal tower of abelian covers will cover $X_N$, so that the number of preimages of the fiber in $X_i$ is at least as large as it is in $X_N$. By Lemma 3.1.2, the torsion homology growth rate is $o(1)$ as $N$ tends to infinity. Thus, (3) implies (2).

Now, suppose that $T_\psi$ admits a non–cyclotomic fibration. Choose a lift $\widetilde{\psi}$ such that $\rho(\widetilde{\psi}) > 1$ on $H_1(\widetilde{S}, \mathbb{C})$. Choose a common finite cover of $T_\psi$ and $T_{\widetilde{\psi}}$. We will abuse notation and denote the cover by $T_\psi$ and the pre–chosen fiber by $S$. If $b_1(T_\psi) = 1$ then the exponential torsion homology growth with respect to any tower of homology covers follows from Proposition 3.2.1.

If $b_1(T_\psi) > 1$ then we have that the invariant cohomology $\mathrm{Hom}(\pi_1(S), \mathbb{Z})^\psi$ is nontrivial. Consider the homology covers $\{S_N\}$ of $S$ which are $\psi$–invariant, namely those which arise from reduction of the $\psi$–coinvariant homology of $S$ modulo $N$. The complex homology of each $S_N$ splits into deck group eigenspaces, one for each irreducible representation $\chi$ of the deck group. The characteristic polynomial of $\psi$ acting on these eigenspaces is given by $\chi(A)$, where $A$ is the Alexander polynomial of $T_\psi$.

Consider the character variety

$$\mathfrak{X} = \mathrm{Hom}(\pi_1(S), S^1)^\psi.$$

Let $\rho$ be the function on $\mathfrak{X}$ which assigns to a representation $\chi$ the modulus of the largest root of $\chi(A)$. Clearly $\rho$ is a continuous function, which achieves at least one local maximum greater than one on $\mathfrak{X}$. It follows that as we range over the eigenvalues of $\widetilde{\psi}_*$ in its action on $H_1(S_N, \mathbb{C})$, for each $\lambda_0 \geq 1$ there is a proportion $q$ such that at least $q$ of the eigenvalues of $\widetilde{\psi}_*$ have modulus at least $\lambda_0$. If $\lambda_0$ is less than $\rho(\widetilde{\psi}_*)$ then $q$ is positive.

Recall that if $A \in GL_n(\mathbb{Z})$ and $\Gamma_A$ is the semidirect product obtained from $\mathbb{Z}$ and $\mathbb{Z}^n$ with the $\mathbb{Z}$–conjugation action given by $A$, then the torsion homology $H_1(\Gamma_A, \mathbb{Z})_{tors}$ is given by $\det'(A - I)$. We can now estimate the torsion homology growth of the homology covers of $T_\psi$. If $b_1(T_\psi) = n$ then the $N^{th}$ homology cover $T_N$ has degree approximately $N^n$ over $T_\psi$. If $T_\psi$ itself has no torsion homology then this estimate is precise. Otherwise, the estimate may be off by a constant factor which is no larger than the size of the torsion homology of $T_\psi$. The proportion of the eigenvalues of $\widetilde{\psi}_*$ acting on $H_1(S_N, \mathbb{C})$ with length at least $\lambda_0$ is $q > 0$. Note that for any complex number $z$, we have $|z - 1| \geq |z| - 1$. It follows that

$$|\det{}'(\widetilde{\psi}_*^N - I)| \geq (\lambda_0^N - 1)^{q \cdot (\operatorname{rk} H_1(S_N, \mathbb{Z}))} \cdot C_N,$$

where $C_N$ is given by

$$\prod |\mu^N - 1|,$$

where $\mu$ ranges over the eigenvalues of $\widetilde{\psi}_*$ whose modulus is less than $\lambda_0$ and which are not equal to 1. The rank of $H_1(S_N, \mathbb{Z})$ is approximately proportional to the degree of $S_N$ over $S$. Since the degree of $S_N$ over $S$ is $N^{n-1}$, we get two possibilities for the value of the rank, corresponding to whether of not $S$ is closed. If $S$ is closed of genus

$g$ then the genus $g_N$ of $S_N$ satisfies

$$(2g_N - 2) = N^{n-1}(2g - 2),$$

so that

$$\operatorname{rk} H_1(S_N, \mathbb{Z}) = N^{n-1}(2g - 2) + 2.$$

Otherwise, if $r$ denotes the rank of $H_1(S, \mathbb{Z})$ then the rank $r_N$ of $H_1(S_N, \mathbb{Z})$ satisfies

$$r_N - 1 = N^{n-1}(r - 1),$$

so that

$$\operatorname{rk} H_1(S_N, \mathbb{Z}) = N^{n-1}(r - 1) + 1.$$

The logarithm of the right hand side of the inequality

$$|\operatorname{det}'(\widetilde{\psi}_*^N - I)| \geq (\lambda_0^N - 1)^{q \cdot (\operatorname{rk} H_1(S_N, \mathbb{Z}))} \cdot C_N$$

is bounded below by

$$|\chi(S)| \cdot q \cdot \log(\lambda_0^N - 1) \cdot N^{n-1} + \log C_N.$$

Thus the logarithm of the order of the torsion homology of $T_N$, normalized by the degree of the cover $T_N \to T_\psi$, is at least

$$\frac{|\chi(S)| \cdot q \cdot \log(\lambda_0^N - 1) \cdot N^{n-1} + \log C_N}{N^n} = |\chi(S)| \cdot q \cdot \frac{\log(\lambda_0^N - 1)}{N} + \frac{\log C_N}{N^n}.$$

Unfortunately, we have little control over the size of

$$\frac{\log C_N}{N^n},$$

and for a given $N$, this quantity may be very negative.

Replace $\widetilde{\psi^N}$ by $\widetilde{\psi^{kN}}$. In the limit as $k$ tends to infinity, only the eigenvalues of norm at least one will contribute to the value of $\log C_{k\cdot N}/(k \cdot N^n)$, so that

$$\lim_{k\to\infty} \frac{\log C_{k\cdot N}}{k \cdot N^n} \geq 0.$$

This is because the eigenvalues occurring in

$$C_{k\cdot N} = \prod |\mu^{k\cdot N} - 1|$$

are the same as in the expansion for $C_N$, but their exponents are now larger.

Thus, we obtain

$$\lim_{k\to\infty} \left( |\chi(S)| \cdot q \cdot \frac{\log(\lambda_0^{k\cdot N} - 1)}{k \cdot N} + \frac{\log C_{k\cdot N}}{k \cdot N^n} \right) \geq |\chi(S)| \cdot q \cdot \log \lambda_0 > 0.$$

Thus, for each homology cover $X_N \to X$ we have produced a finite abelian cover $X_i \to X$ which covers $X_N$ and which satisfies

$$\frac{\log |H_1(X_i, \mathbb{Z})_{tors}|}{\deg(X_i \to X)} \geq |\chi(S)| \cdot \frac{q}{2} \cdot \log \lambda_0.$$

Since for any finite abelian cover $X_A$ of $X$ there is a homology cover of $X$ which covers $X_A$, it is clear that the desired universal tower of abelian covers exists. Therefore, (1) implies (3). $\square$

We remark briefly on the role of the multiplicative constant $|\chi(S)|$ in the estimate above. On the one hand, the torsion homology growth exponent of a 3–manifold along towers of homology covers is independent of the choice of fibration. On the other hand, it is often possible to choose many different fibrations of a given hyperbolic 3–manifold, and the associated monodromies may have expansion factors tending to one and the Euler characteristics of the fibers tend to $-\infty$. Since the expansion factor provides an upper bound for $\lambda_0$, rescaling by $|\chi(S)|$ prevents our estimate from vanishing.

Corollary 1.3.5 follows more or less immediately now. Recall its statement:

**Corollary 3.3.2.** *Let $\psi$ be a pseudo-Anosov mapping class. There exists a lift $\widetilde{\psi}$ to a finite cover satisfying $\rho(\widetilde{\psi}_*) > 1$ if and only if for each mapping class $\phi$ which is commensurable to $\psi$, there exists a lift $\widetilde{\phi}$ to a finite cover satisfying $\rho(\widetilde{\phi}_*) > 1$.*

*Proof.* Suppose $\phi$ is the monodromy of a bundle which is commensurable with $T_\psi$. By assumption there is a common finite cover $X$ of $T_\phi$ and $T_\psi$ which has exponential torsion homology growth with respect to some universal tower of abelian covers. It follows that $\phi$ cannot be the monodromy of a cyclotomic bundle. The converse is immediate. $\square$

## 3.4 Corollary 1.3.5 for the braid $\sigma_1\sigma_2^{-1}$

Let $S$ be a disk with three punctures. The braid $\sigma_1\sigma_2^{-1}$ induces a pseudo-Anosov mapping class of $S$, which we call $\psi = \psi_{(1,0)}$. We have that $H^1(T_\psi, \mathbb{Z}) \cong \mathbb{Z}^2$, which we write as $\langle u, x\rangle$. The cohomology class $u$ is associated to the monodromy $\psi$. For all primitive pairs $(b, a)$ with $|b| > |a|$, the cohomology class given by sending $u$ to $b$ and $x$ to $a$ gives another fibration of $T_\psi$, and we call the associated monodromy $\psi_{(a,b)}$. We summarize some facts about the mapping classes $\{\psi_{(a,b)}\}$ which can be found in [17]:

**Proposition 3.4.1.** *Let $\psi$ be the pseudo-Anosov mapping class associated to $\sigma_1\sigma_2^{-1}$, and let $\psi_{(a,b)}$ be the monodromy associated to the cohomology class $(a, b)$.*

1. *The Teichmüller polynomial of $T_\psi$ is given by*

$$\Theta(u,x) = u^2 - u(1 + x + x^{-1}) + 1.$$

2. *The Alexander polynomial of $T_\psi$ is given by*

$$\Delta(u,x) = u^2 - u(1 - x - x^{-1}) + 1.$$

3. *Let $\mathcal{F}_{a,b}$ be the $\psi_{(a,b)}$ invariant foliation on the fiber associated to that monodomy. Then $\mathcal{F}_{a,b}$ has no interior singularities.*

Corollary 1.3.5 can be verified almost immediately for $\psi$ from the previous proposition. One way to see that each $\psi_{(a,b)}$ has a lift $\widetilde{\psi_{(a,b)}}$ satisfying $\rho(\widetilde{\psi_{(a,b)}}) > 1$ is to observe that part 3 of the previous proposition implies that each $\mathcal{F}_{a,b}$ becomes orientable on a finite cover of the fiber. It follows that the homological spectral radius and the expansion factor coincide on such a cover, and therefore the homological spectral radius is greater than one. Alternatively, there is a finite cover of the base surface (given by sending $x$ to $-1$) where the invariant foliation $\mathcal{F}$ of $\psi$ becomes orientable, which can be seen from the fact that the Teichmüller and Alexander polynomial coincide for that cover. It follows that the suspension of the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ by the corresponding lift $\widetilde{\psi}$ of $\psi$ is orientable. It follows that every other fibration of $T_{\widetilde{\psi}}$ in the fibered face corresponding to $\widetilde{\psi}$ has an orientable invariant foliation. Then, the homological spectral radius and expansion factor must coincide.

# Chapter 4

# Homological actions on finite covers and the Nielsen–Thurston classification

In this Chapter, we will prove Theorems 1.2.3 and 1.6.1.

## 4.1 Nontriviality of the homological action

We will first give a proof of the following result:

**Theorem 4.1.1** (cf. Theorem 1.2.3)**.** *For each nontrivial mapping class $\psi$, there is a finite cover $\widetilde{S}$ of $S$ such that each lift of $\psi$ to $\mathrm{Mod}(\widetilde{S})$ acts nontrivially on $H_1(\widetilde{S},\mathbb{C})$. If $\{\widetilde{S}_i\}$ is any exhausting collection of finite characteristic covers of $S$ with respect to which $\pi_1(S)$ is conjugacy separable, then we may take $\widetilde{S}$ to be $\widetilde{S}_i$ for some $i$.*

Recall $\{\widetilde{S}_i\}$ is exhausting if

$$\bigcap \pi_1(\widetilde{S}_i) = \{1\}.$$

The following analogue for free group automorphisms will follow immediately:

**Theorem 4.1.2.** *For each nontrivial $\psi \in \mathrm{Out}(F_n)$ there exists a finite index $H < F_n$ such that each lift of $\psi$ to $\mathrm{Out}(H)$ acts nontrivially on $H_1(H, \mathbb{C})$. If $\{H_i\}$ is any exhausting collection of finite characteristic subgroups of $F_n$ with respect to which $F_n$ is conjugacy separable, then we may take $H$ to be $H_i$ for some $i$.*

We need one more result before giving the proof:

**Lemma 4.1.3.** *Let $G$ be a nonabelian free group or surface group, and let $G'$ be a finite index normal subgroup with $G/G' \cong \Gamma$. Then $\Gamma$ acts faithfully on $H_1(G', \mathbb{C})$.*

*Proof.* This follows from [7] or [20]. When $G = \pi_1(S_g)$ then $H_1(G', \mathbb{C})$ is $2g-2$ copies of the regular representation and 2 copies of the trivial representation. When $G = F_n$ then $H_1(G', \mathbb{C})$ is $n-1$ copies of the regular representation and one copy of the trivial representation. □

*Proof of Theorem 4.1.1.* Let $\psi$ be a nontrivial mapping class. Then there is a $g \in \pi_1(S)$ such that $\psi(g)$ is not conjugate to $g$ and a finite Galois cover $S_i$ such that the images of $g$ and $\psi(g)$ are not conjugate in

$$G = \pi_1(S)/\pi_1(S_i).$$

Then there is a character $\chi$ of $G$ for which $\chi(g) \neq \chi(\psi(g)) = \psi(\chi)(g)$. But then any lift of $\psi$ sends the $\chi$–isotypic part of $H_1(S_i, \mathbb{C})$ to the $\psi(\chi)$–isotypic part of $H_1(S_i, \mathbb{C})$, so that each lift of $\psi$ acts nontrivially on $H_1(S_i, \mathbb{C})$. □

It is clear from the proof of Theorem 4.1.1 that the most general statement is:

**Theorem 4.1.4.** *Let* $\{\widetilde{S}_i\}$ *be any exhausting sequence of finite characteristic covers of* $S$ *with respect to which* $\pi_1(S)$ *is conjugacy separable. Then for each nontrivial* $\psi \in \mathrm{Mod}(S)$*, there is an* $i$ *such that each lift* $\widetilde{\psi}$ *acts nontrivially on* $H_1(\widetilde{S}_i, \mathbb{C})$*.*

For certain mapping classes one can explicitly exhibit finite covers of $S$ to which these mapping classes lift and act nontrivially on the integral homology of the cover. We can do this explicitly for any Dehn twist, and the idea is identical to the lifting of separating curves to nonseparating ones:

**Proposition 4.1.5.** *Let* $T \in \mathrm{Mod}(S)$ *be a Dehn twist. Then there is a two–fold lift* $\widetilde{T}$ *which acts with infinite order on* $H_1(\widetilde{S}, \mathbb{C})$*.*

*Proof.* If $T$ is a twist about a nonseparating curve, there is nothing to show. If $T$ is a twist about a separating curve $\gamma$ then $\gamma$ splits $S$ into subsurfaces $S_1$ and $S_2$. Let $\alpha$ and $\beta$ be nonseparating simple closed curves of $S$ which are contained in $S_1$ and $S_2$ respectively, and extend their homology classes to a basis for $H_1(S, \mathbb{Z})$. Define a map

$$H_1(S, \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$$

which kills all the basis elements other than $\alpha$ and $\beta$ and which sends those two homology classes to the nontrivial element. On the resulting cover $\widetilde{S}$, the curve $\gamma$ will have two lifts and $T$ lifts to a simultaneous twist about the two components. One can check easily that this twist has infinite order as an automorphism of $H_1(\widetilde{S}, \mathbb{C})$. □

## 4.2 The Nielsen-Thurston classification of mapping classes and homology

We are now in a position to give the proof of Theorem 1.6.1.

**Lemma 4.2.1.** *Let $g \in \pi_1(S)$ be a nonperipheral nonidentity element, and suppose that $\psi \in \mathrm{Mod}(S)$ has infinite order and preserves the conjugacy class of $g$. Then $\psi$ is reducible.*

*Proof.* Let $\gamma$ be a geodesic representative of the free homotopy class of $g$. Fixing a hyperbolic metric on $S$, we consider the length of the geodesic representative of $\psi^n(\gamma)$ as a function of $n$. If $\psi$ is pseudo-Anosov, we have that this function grows like $\lambda^n$, where $\lambda$ is the expansion factor of $\psi$ (see [12], exposé 10 by Fathi and Shub). If $\psi$ preserves the conjugacy class of $g$ then this function is constant, a contradiction. □

*Proof of Theorem 1.6.1.* Recall that for each characteristic quotient $G$ of $\pi_1(S)$ with corresponding cover $\widetilde{S}$, we have a map

$$\tau_G : \mathrm{Mod}(S) \to \mathrm{Sym}(X(G)).$$

If $\psi$ has finite order, then we may assume $\psi^N = 1$. Then for each quotient $G$, the image of $\tau_G(\psi)$ has order at most $N$. Thus as an automorphism of the $G$–characters occurring in $H_1(\widetilde{S}, \mathbb{C})$, we have

$$\sup_G |\tau_G(\psi)| < \infty.$$

Suppose that $1 \neq g \in \pi_1(S)$ is not freely homotopic to a puncture and that its conjugacy class is preserved by $\psi$, possibly after passing to a power. Then for each finite characteristic quotient $G$ and each character $\chi \in X(G)$ we have

$$\tau_G(\psi)(\chi)(g) = \chi(\psi(g)) = \chi(g),$$

where here by $\psi(g)$ we mean that we are choosing any lift of $\psi$ to $\mathrm{Aut}(\pi_1(S))$ and applying it to $g$.

Suppose that for each such nonperipheral $g$, we have that the conjugacy class of $g$ is not preserved by $\psi$. Then there is a finite characteristic $G$ such that the conjugacy classes $g$ and $\psi(g)$ are not equal, where again by $\psi(g)$ we mean that we are choosing any lift of $\psi$ to $\mathrm{Aut}(\pi_1(S))$. Therefore we have some character $\chi \in X(G)$ such that

$$\chi(g) \neq \tau_G(\psi)(\chi)(g).$$

This completes the characterization of finite order, reducible and pseudo-Anosov mapping classes, respectively. □

We remark that whereas a finite order mapping class $\psi$ has finite order under $\tau_G$ for each $G$, an arbitrary lift of $\psi$ to an automorphism of $H_1(\widetilde{S}, \mathbb{C})$ may acquire arbitrarily high order, since the lift is defined up to pre– and post–composition with elements of $G$.

## 4.3 Free groups and detecting the classification of free group automorphisms

Analogously to the Nielsen–Thurston classification, there is a classification of free group automorphisms. An (outer) automorphism $\phi$ of the free group on $n$ generators

$F_n$ is called either **finite order**, **reducible** or **irreducible**. The reader who is not familiar with this classification seeking an accessible introduction to the classification should consult Bestvina's article [4].

We will need to appeal to the following characterization of reducible automorphisms which can be found in [5]:

**Lemma 4.3.1.** *Let $\phi \in Out(F_n)$. Then $\phi$ is reducible if and only if there are free factors $F_{n_i}$, $1 \leq i \leq k$, $n_1 < n$, such that $F_{n_1} * \cdots * F_{n_k}$ is a free factor of $F_n$ and $\phi$ cyclically permutes the conjugacy classes of the $\{F_{n_i}\}$.*

Thus if $\phi$ is reducible, there is a power of $\phi$ which preserves the conjugacy class of a free factor of $F_n$.

Let $\phi \in \mathrm{Out}(F_n)$ and let $\Phi$ be any lift of $\phi$ to $\mathrm{Aut}(F_n)$. If $H < F_n$ is a finite index, normal, $\Phi$–invariant subgroup of $F_n$, then $\Phi$ can be viewed as an element of $\mathrm{Aut}(H)$. It can then be projected to $\mathrm{Out}(H)$. If $\phi \in \mathrm{Out}(F_n)$, we will write $\widetilde{\phi}$ for the restriction of some lift $\Phi$ to $\mathrm{Out}(H)$. With terminology analogous to mapping class groups, we will call $\widetilde{\phi}$ a **lift** of $\phi$. The ambiguity of $\widetilde{\phi}$ is exactly analogous to the ambiguity of lifts of mapping classes.

The first result about automorphisms of free groups is the following result, whose proof is exactly the same as Theorem 1.2.3:

**Theorem 4.3.2.** *Let $1 \neq \phi \in \mathrm{Out}(F_n)$. There exists a finite index, normal subgroup $H$ of $F_n$ such that each lift $\widetilde{\phi}$ to $\mathrm{Out}(H)$ acts nontrivially on $H^{ab}$.*

We now show how to use abelianizations of finite index subgroups of $F_n$ to detect the Nielsen–Thurston classification of $\phi$. Suppose $G$ is a finite quotient of $F_n$ with kernel $H$. As in the surface group case, we obtain an action of $G$ on $H^{ab}$, and the $G$–representation $H^{ab} \otimes \mathbb{C}$ consists of $n-1$ copies of the regular representation of $G$ together with one copy of the trivial representation. Now assume that $G$ is a characteristic quotient of $F_n$. If $X(G)$ denotes the set of irreducible characters of $G$, we obtain a map

$$\tau_G : \mathrm{Out}(F_n) \to \mathrm{Sym}(X(G)).$$

We will need the following fact in order to analyze the case of reducible automorphisms:

**Lemma 4.3.3** ( [28], Proposition 3.10)**.** *Let $K < F_n$ be a finitely generated subgroup of $F_n$. Then there is a finite index subgroup $H$ of $F_n$ such that $K$ is a free factor of $H$.*

**Theorem 4.3.4.** *Let $1 \neq \phi \in \mathrm{Out}(F_n)$.*

1. *The automorphism $\phi$ is finite if and only if*

$$\sup_G |\tau_G(\phi)| < \infty.$$

2. *The automorphism $\phi$ is reducible if and only if there is an $n > 0$ and two nontrivial, finitely generated subgroups $K_1, K_2 < F_n$ which together generate $F_n$, such that $K_1 \cap K_2 = \{1\}$ and such that for each $k_i \in K_i$ there is a $k_i' \in K_i$ satisfying*

$$\tau_G(\phi^n)(\chi)(k_i) = \chi(k_i')$$

*for each characteristic quotient $G$ and each $\chi \in X(G)$.*

3. *The automorphism $\phi$ is irreducible if and only if both (1) and (2) fail.*

*Proof.* The characterization of finite order automorphisms is clear, since if $\phi$ preserves the conjugacy class of each element of $F_n$ then it is inner. Suppose that $\phi$ is reducible. After passing to a power of $\phi$, there is a nontrivial free factor decomposition

$$K_1 * \cdots * K_m$$

of $F_n$ which is preserved by $\phi$ up to conjugacy. Thus for any lift of $\phi$ to $\mathrm{Aut}(F_n)$, we have that for each $k_i \in K_i$, $\phi(k_i)$ is conjugate to an element of $K_i$. This condition (2) holds.

Conversely, suppose that condition (2) holds. Then $K_1$ and $K_2$ are a free factor decomposition of $F_n$, and the conjugacy classes of the factors are preserved by a power of $\phi$. $\square$

# Chapter 5

# Large groups

In this chapter we will relate homological actions of mapping classes and 3–manifold theory, showing that (M2) and (S2) cannot both fail to hold for a mapping class $\psi$:

**Theorem 5.0.5.** *Suppose $\psi \in \mathrm{Mod}(S)$ has $\rho(\psi_*) = 1$. Then at least one of the following conclusions holds:*

1. *There is a lift $\widetilde{\psi}$ to a finite abelian cover such that $\rho(\widetilde{\psi}_*) > 1$.*

2. *The mapping torus $T_\psi$ is large.*

In order to prove Theorem 5.0.5, we will show that it satisfies condition (2) of the following more general result (cf. Theorem 1.5.1):

**Theorem 5.0.6.** *Let $X$ be a finite CW complex. The following are equivalent:*

1. *The $X$ is large.*

2. *There exists a finite cover $Y$ of $X$ such that for each $N$, there is a finite abelian cover $Y_N$ of $Y$ with $b_1(Y_N) \geq N$.*

We noted before that for any finitely presented group $G$, largeness implies virtually infinite first Betti number. Theorem 1.5.1 gives us the strongest possible reverse implication.

## 5.1 Large groups

In order to show that certain groups are large, we will use the work of M. Lackenby in [21]. The relevant tools are **homology rank gradient** and **property ($\tau$)**. To define these, we fix a prime $p$ and a sequence of nested finite index normal subgroups $\{\Gamma_i\}$ of a fixed group $\Gamma$. We write $d(\Gamma_i)$ for the dimension of $H_1(\Gamma_i, \mathbb{Z}/p\mathbb{Z})$ as a vector space over $\mathbb{Z}/p\mathbb{Z}$. The homology rank gradient of the sequence $\{\Gamma_i\}$ is

$$\gamma = \inf_i \frac{d(\Gamma_i)}{[\Gamma : \Gamma_i]}.$$

When $\gamma > 0$, we say that the $\{G_i\}$ have **positive modulo $p$ homology rank gradient**.

If $G$ is a finitely generated group with a finite generating set $S$ and $\{G_i\}$ is a collection of finite index subgroups, we write $X(G/G_i, S)$ for the coset graph of $G_i$ in $G$. If $A$ is a set of vertices $V$ in a graph, we write $\partial A$ for the set of edges with exactly one vertex in $A$. The **Cheeger constant** $h(X)$ of a finite graph $X$ is given by

$$h(X) = \min_{A \subset V,\, 0 < |A| \leq |V|/2} \left\{ \frac{|\partial A|}{|A|} \right\}.$$

$G$ has property ($\tau$) with respect to $\{G_i\}$ if the infimum of $h(X(G/G_i, S))$ is positive for some initial choice of $S$. Lubotzky's book [25] contains a detailed exposition on property ($\tau$).

Lackenby has proved many different results concerning conditions under which a group is large. The one we will cite here can be found in [21]:

**Theorem 5.1.1.** *Let $G$ be a finitely presented group. Then the following are equivalent:*

1. *$G$ is large.*

2. *There exists a sequence of proper nested finite index subgroups*

$$G > G_1 > G_2 > \cdots$$

   *and a prime $p$ such that:*

   (a) *$G_{i+1}$ is normal in $G_i$ and has index a power of $p$ in $G_i$.*

   (b) *$G$ does not have property $(\tau)$ with respect to $\{G_i\}$.*

   (c) *$\{G_i\}$ has positive modulo $p$ homology rank gradient.*

We begin by ruling out property $(\tau)$ for our purposes.

**Lemma 5.1.2.** *Let $G$ be finitely generated by a set $S$, and suppose that for each $i$ we have that $G/G_i$ is abelian. Then $G$ does not have property $(\tau)$ with respect to $\{G_i\}$.*

*Proof.* Write

$$G_\infty = \bigcap_i G_i.$$

We will show that the Cheeger constant will arbitrarily small as $i$ tends to infinity. Note that in a Cayley graph for $G/G_\infty$ with respect to $S$, we may define the Cheeger constants of finite subgraphs by looking at the ratio of the size of the boundary of a finite set to the size of the set itself. Since $G/[G,G]$ is amenable, the infimum of these Cheeger constants will be zero. Furthermore, note that since the intersection of the subgroups $\{G_i\}$ is $G_\infty$, for any finite subset of the vertices in a Cayley graph for $G/G_\infty$ we may find an $i$ so that this set of vertices is mapped injectively to the set of vertices for the Cayley graph for $G/G_i$ with respect to $S$. The degree of each vertex is non–increasing as we project the Cayley graph of $G/G_\infty$ to the Cayley graph of $G/G_i$. It follows that the number of vertices in the boundary of a finite set of vertices in the Cayley graph of $G/G_\infty$ cannot increase under the projection map. It follows that $\inf h(X(G, G_i, S)) = 0$. □

We thus obtain a corollary to Lackenby's result by combining Lemma 5.1.2 and the fact that for any finitely generated group $G$ we have

$$\operatorname{rk} H_1(G, \mathbb{Z}) \leq \operatorname{rk} H_1(G, \mathbb{Z}/p\mathbb{Z}) :$$

**Corollary 5.1.3.** *Let $G$ be a finitely presented group, and let $\{G_i\}$ be a tower of nested, normal, $p$–power index subgroups of $G$ such that $G/G_i$ is abelian for each $i$. Suppose that*

$$\inf_i \frac{\operatorname{rk} H_1(G_i, \mathbb{Z})}{[G : G_i]} > 0.$$

*Then $G$ is large.*

We can now establish some facts about the homology growth of fibered 3–manifolds. The homology growth of fibered 3–manifold groups is very closely related to the action of mapping classes on the homology of finite covers of a surface. Recall that if

$$S \to M \to S^1$$

is a fibered 3–manifold with monodromy $\psi$, the homology $H_1(M,\mathbb{Z})$ is given by $\mathbb{Z}\oplus F$, where $F$ is the homology of $S$ which is co–invariant under the action of $\psi$.

Let $\widetilde{S} \to S$ be a finite characteristic cover and let $\psi \in \mathrm{Mod}(S)$, and suppose that $\psi$ commutes with the Galois group $G$ of the cover. We obtain an induced covering $M' \to M$ as follows: the fundamental group of $M$ is given by a semidirect product

$$1 \to \pi_1(S) \to \pi_1(M) \to \mathbb{Z} \to 1.$$

Since the Galois group $G$ commutes with the action of $\psi$, we obtain a homomorphism $\pi_1(M) \to G \times \mathbb{Z}$, and by composing with the projection onto the first factor, a homomorphism $\pi_1(M) \to G$.

Note that the homology contribution of the base space $S^1$ is in the kernel of this homomorphism. In particular, we obtain a covering map $M' \to M$ which lifts the identity map $S^1 \to S^1$. Furthermore, $\pi_1(M')$ fits into a short exact sequence

$$1 \to \pi_1(\widetilde{S}) \to \pi_1(M') \to \mathbb{Z} \to 1,$$

where the conjugation action of $\mathbb{Z}$ on $\pi_1(\widetilde{S})$ is the restriction of the action of $\psi$. It follows that the rank of the homology of $M'$ is given by the rank of homology co–invariants of the action of $\psi$ on $H_1(\widetilde{S},\mathbb{Z})$ (equivalently the rank of the $\psi$–invariants of $H^1(\widetilde{S},\mathbb{Z})$).

We can now establish a fact about Magnus kernels before providing a full proof of Theorem 5.0.5. Let $S^{ab}$ be the universal abelian cover of $S$. Recall that the **Magnus kernel** is the subgroup of the mapping class group which acts trivially on $H_1(S^{ab},\mathbb{Z})$. It is not a priori clear that the Magnus kernel is nontrivial, but in fact it is infinitely generated (see [8]).

We first need to make the following observation:

**Lemma 5.1.4.** *Let $\psi \in \mathrm{Mod}(S)$ be a mapping class contained in the Magnus kernel. Then $\psi$ acts trivially on the integral homology of each finite abelian cover of $S$.*

*Proof.* Since $\psi$ is in the Magnus kernel, then as an automorphism of $\pi_1(S)$, $\psi$ acts trivially on the universal metabelian quotient

$$\pi_1(S)/[[\pi_1(S), \pi_1(S)], [\pi_1(S), \pi_1(S)]].$$

Suppose that $\widetilde{S} \to S$ is a finite abelian cover with Galois group $A$. Then we have a short exact sequence

$$1 \to H_1(\widetilde{S}, \mathbb{Z}) \to M \to A \to 1.$$

Since $M$ is evidently metabelian, it is a quotient of

$$\pi_1(S)/[[\pi_1(S), \pi_1(S)], [\pi_1(S), \pi_1(S)]].$$

It follows that $\psi$ restricts to the identity on $H_1(\widetilde{S}, \mathbb{Z})$. $\square$

**Proposition 5.1.5.** *Suppose that $\psi$ is contained in the Magnus kernel. Then the fundamental group of the manifold $M_\psi$ is large.*

*Proof.* We suppose that $S$ is closed. The proof in the non–closed case is analogous. Let $p$ be any prime, $g > 1$ the genus of $S$, and $H_i$ be the kernel of the map

$$\pi_1(S) \to H_1(S, \mathbb{Z}/p^i\mathbb{Z}).$$

It is clear that $\{H_i\}$ forms a sequence of subgroups such that $\pi_1(S)/H_i$ is an abelian $p$–group for each $i$. Furthermore, the rank of $H_1(S, \mathbb{Z}/p^i\mathbb{Z})$ is $p^{2gi}$. By an Euler character argument, we have that the genus of the $i^{th}$ surface $S_i$ corresponding to $H_i$ is $p^{2gi}(g-1)+1$. It follows that the rank of $H_i^{ab}$ is $p^{2gi}(2g-2)+2$.

Now let $\psi$ be any mapping class in the Magnus kernel and let $T_\psi$ be suspension of $\psi$ with fundamental group $G$. Since $\psi$ acts trivially on $H_1(S,\mathbb{Z})$, we have that $\psi$ commutes with $\pi_1(S)/H_i$ for each $i$ so that each $H_i$ gives us a finite cover $T_i$ of $T_\psi$ with fundamental group $G_i$. Topologically, $T_i$ is just the suspension of $\psi$ as a mapping class of $S_i$. Since $\psi$ is in the Magnus kernel, it follows that the rank of $G_i^{ab}$ is $p^{2gi}(2g-2)+2$. Note furthermore that the index $[G : G_i]$ is equal to $[\pi_1(S) : H_i]$, namely $p^{2gi}$. Since $g > 1$, the ratio between the rank of the homology of $G_i$ and $[G : G_i]$ is bounded away from zero. Since $G/G_i$ is abelian for all $i$, Corollary 5.1.3 implies that $G$ is large. □

## 5.2 Alexander varieties and largeness

We begin by relating the unbounded growth of $b_1$ over finite covers to some aspects of the Alexander variety.

**Lemma 5.2.1.** *Let $X$ be a finite CW complex and suppose that for each $N$, there is a finite abelian cover $X_N$ of $X$ with $b_1(X_N) \geq N$. Then the Alexander variety $V_1(\pi_1(X))$ contains infinitely many torsion points.*

*Proof.* Recall that if $\alpha : \pi_1(X) \to \Gamma$ is a surjection to a finite abelian group, then the first Betti number of the associated cover $X_\alpha$ satisfies

$$b_1(X_\alpha) = b_1(X) + \sum_{i=1}^{r} |V_i(\pi_1(X)) \cap \widehat{\alpha}(\widehat{\Gamma} \setminus \widehat{1})|$$

(see Section 2.8). Since $b_1(X_\alpha)$ can be made arbitrarily large as $\alpha$ varies over all finite abelian quotients of $\pi_1(X)$ and since we have a sequence of inclusions

$$V_r(\pi_1(X) \subset \cdots \subset V_1(\pi_1(X)),$$

it follows that for each $N$ there is a finite abelian quotient

$$\alpha_N : \pi_1(X) \to \Gamma_N$$

for which

$$|V_1(\pi_1(X)) \cap \widehat{\alpha_N}(\widehat{\Gamma_N} \setminus \widehat{1})| \geq N.$$

Since each element of $\widehat{\alpha_N}(\widehat{\Gamma_N} \setminus \widehat{1})$ is a torsion point in $V_1$, the lemma follows. $\square$

An immediate consequence of Laurent's Theorem and Lemma 5.2.1 is that there is a rational plane $P \subset V_1(\pi_1(X))$ whose intersection with $(S^1)^r$ has positive dimension.

**Lemma 5.2.2.** *Let $X$ be a finite CW complex and let $V_1(\pi_1(X)) \subset (\mathbb{C}^*)^r$ be its Alexander variety. Suppose there is a rational plane $P \subset V_1(\pi_1(X))$ whose intersection with*

$$(S^1)^r \subset (\mathbb{C}^*)^r$$

*has positive dimension. Then $\pi_1(X)$ is large.*

*Proof.* Clearly we may suppose that at least one component of $V_1(\pi_1(X))$ has positive dimension. Since $V_1(\pi_1(X))$ sits inside of $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$, we may suppose that $b_1(X) \geq 1$. Write $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^m \oplus A$, where $m > 1$ and where $A$ is a finite abelian group. Choose a basis $\{w_1, \ldots, w_m\}$ for $H^1(X, \mathbb{Z}) = \mathrm{Hom}(\pi_1(X), \mathbb{Z})$. Composition of integral cohomology classes of $X$ with elements of $\mathrm{Hom}(\mathbb{Z}, \mathbb{C}^*)$ associates to each basis element $w_i$ a distinguished copy of $\mathbb{C}^*$ inside of $(\mathbb{C}^*)^m$. Now note that $\widehat{A} \cong A$, and that $\widehat{A}$ consists of a finite set of points in $(S^1)^{r-m}$. Thus, we may think of $\mathrm{Hom}(\pi_1(X), \mathbb{C}^*)$ as sitting inside of

$$(\mathbb{C}^*)^m \times \widehat{A} \subset (\mathbb{C}^*)^r.$$

We will write $\{u_1, \ldots, u_m\}$ for the dual basis of $H_1(X, \mathbb{Z})/A$ determined by $\{w_1, \ldots, w_m\}$. Write

$$A = \mathbb{Z}/n_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/n_{r-m}\mathbb{Z},$$

where $n_i | n_{i+1}$, and let $\{x_1, \ldots, x_{r-m}\}$ be each identified with the corresponding generator $1 \in \mathbb{Z}/n_i\mathbb{Z}$.

Choose a rational splitting $(\mathbb{C}^*)^r$ which is compatible with the choices in the previous paragraph. Since $P \cap (S^1)^r$ has positive dimension, there is a rational, one–dimensional subtorus of $(S^1)^r$ which is also contained in $V_1(\pi_1(X))$. Such a subtorus is given (with respect to the chosen splitting) by a map $\mathbb{R} \to (S^1)^r \subset (\mathbb{C}^*)^r$ of the form

$$\gamma : t \mapsto (\exp(2\pi i(a_1 t + s_1)), \ldots, \exp(2\pi i(a_m t + s_m)), \exp(2\pi i s_{m+1}), \ldots, \exp(2\pi i s_r)),$$

where $\{a_1, \ldots, a_m\}$ are integers which are not simultaneously zero, and where $\{s_1, \ldots, s_r\}$ are rational. Consider the subgroup $\Gamma$ of $S^1$ generated by the roots of unity $\{\exp(2\pi i s_i)\}$, which has order $B$. Choose an identification of the group $\Gamma$ with $\mathbb{Z}/B\mathbb{Z}$, and write $d_i \pmod B$ for the image of $\exp(2\pi i s_i)$ under this identification.

Choose a prime $p$ which divides none of the nonzero $\{a_i\}$ nor $B$. For each $n$, we consider the homomorphism $\alpha_n : \pi_1(X) \to \mathbb{Z}/p^n B\mathbb{Z}$ given by

$$u_i \mapsto a_i + p^n d_i \pmod{p^n B}$$

and by

$$x_j \mapsto p^n d_j \pmod{p^n B}.$$

Let $\zeta$ be a primitive $p^n B^{th}$ root of unity, viewed as an element of $\widehat{\mathbb{Z}/p^n B\mathbb{Z}}$. The image of $\zeta$ under $\widehat{\alpha_n}$ is just

$$(\zeta^{a_1+p^n d_1}, \ldots, \zeta^{a_m+p^n d_m}, \zeta^{p^n d_{m+1}}, \ldots, \zeta^{p^n d_r}).$$

Since $\zeta$ is a primitive $p^n B^{th}$ root of unity, $\zeta^{p^n}$ is a primitive $B^{th}$ root of unity. Whenever $\zeta^{p^n}$ is equal to the generator $1 \in \mathbb{Z}/B\mathbb{Z}$ via the inverse of the identification of $\Gamma$ with $\mathbb{Z}/B\mathbb{Z}$, we have that $\zeta^{p^n d_i}$ is equal to the root of unity $\exp(2\pi i s_i)$. There are $B$ distinct $B^{th}$ roots of unity, so for at least $1/B$ of the primitive $p^n B^{th}$ roots of unity, we will have that $\zeta^{p^n d_i}$ is equal to $\exp(2\pi i s_i)$. Whenever this is the case, we have that the point

$$(\zeta^{a_1+p^n d_1}, \ldots, \zeta^{a_m+p^n d_m}, \zeta^{p^n d_{m+1}}, \ldots, \zeta^{p^n d_r})$$

lies on the subtorus $\gamma$. Writing $X_n$ for the cover corresponding to $\alpha_n$, the formula

$$b_1(X_\alpha) = b_1(X) + \sum_{i=1}^{r} |V_i(\pi_1(X)) \cap \widehat{\alpha}(\widehat{\Gamma} \setminus \widehat{1})|$$

implies that

$$b_1(X_n) \geq \frac{1}{B} p^n B - 1 = p^n - 1.$$

Let $k$ be the multiplicative order of $p$ modulo $B$. We claim that $X_{n+k}$ covers $X_n$. Since $B$ and $p$ are relatively prime, we have a splitting for each $n$:

$$\mathbb{Z}/p^n B \cong \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z},$$

and we have projections $q_1$ and $q_2$ onto the two factors. The map $\alpha_{n+k}$ is given by taking

$$u_i \mapsto a_i + p^{n+k} d_i \pmod{p^{n+k} B}$$

and

$$x_j \mapsto p^{n+k} d_j \pmod{p^{n+k} B}.$$

Considering the images of $\{u_i\}$ and $\{x_j\}$ in

$$\mathbb{Z}/p^n B \cong \mathbb{Z}/p^n\mathbb{Z} \oplus \mathbb{Z}/B\mathbb{Z},$$

we see that the image of $p^k(p^n d_i)$ is equal to the image of $p^n d_i$. Indeed, $d_i$ can be written as $(q_1(d_i), q_2(d_i))$. Since the first coordinate becomes zero when multiplied by $p^n$, we see that

$$p^{n+k}(q_1(d_i), q_2(d_i)) = (0, p^{n+k} q_2(d_i)) = (0, p^n q_2(d_i)) = p^n(q_1(d_i), q_2(d_i)).$$

It follows that $\alpha_n$ is equal to the reduction of $\alpha_{n+k}$ modulo $p^n B$.

Thus, we have a tower of covers

$$\cdots \to X_{2k} \to X_k \to X,$$

where $X_k \to X$ has degree $p^k B$, and where $X_{(n+1)k} \to X_{nk}$ has degree $p^k$ when $n > 0$. In particular $\{X_{nk}\}_{n>0}$ is a tower of abelian $p$–power covers with

$$b_1(X_{nk}) \geq \frac{1}{B} p^{nk} B - 1 = p^{nk} - 1.$$

It follows that this tower has positive modulo $p$ homology rank gradient, so that by Corollary 5.1.3, $X$ is large. □

Finally, we may give a proof of the characterization of large groups:

*Proof of Theorem 1.5.1.* Let $\Gamma$ be a large group. Then there is a finite index subgroup $\Gamma' < \Gamma$ which surjects to the nonabelian free group $F_2$. Choosing a sequence of surjective homomorphisms

$$\phi_N : F_2 \to \mathbb{Z} \to \mathbb{Z}/N\mathbb{Z},$$

we have that $b_1(ker(\phi_N)) \geq N+1$. Precomposing $\phi_N$ with a surjection $\Gamma' \to F_2$, it follows that for each $N$, there is a finite index subgroup $\Gamma'_N < \Gamma$ such that $\Gamma'/\Gamma'_N$ is abelian and such that $b_1(\Gamma'_N) \geq N+1$. Thus (1) implies (2). (2) implies (3) is a consequence of Lemma 5.2.1 and the remark immediately following the proof. (3) implies (1) is a consequence of Lemma 5.2.2. □

We can now prove Theorem 1.4.3:

**Theorem 5.2.3.** *Let $\psi$ be a pseudo-Anosov mapping class, and suppose that for each lift $\widetilde{\psi}$ to a finite cover $\widetilde{S}$, we have $\rho(\widetilde{\psi}_*) = 1$. Then the mapping torus $T_\psi$ is large.*

*Proof.* Let $S$ be the fiber of the fibration of $T_\psi$ with monodromy $\psi$. Since $\rho(\psi_*) = 1$, there is a $k > 0$ such that $b_1(T_{\psi^k}) \geq 2$. Equivalently, $\mathrm{Hom}(\pi_1(S), \mathbb{Z})^{\psi^k}$ is nontrivial. Fix a $\psi^k$–invariant cohomology class $\phi$, and let $\phi_N$ be the map $\pi_1(S) \to \mathbb{Z}/N\mathbb{Z}$ given by reducing the image of $\phi$ modulo $N$. For each $N$, we will write $S_N$ for the corresponding cover of $S$. For each irreducible character $\chi$ of $\mathbb{Z}/N\mathbb{Z}$, we may consider the $\chi$–eigenspace of $H_1(S_N, \mathbb{C})$, which we can identify with the twisted homology space $H_1(S, \mathbb{C}_\chi)$. Since $\rho(\widetilde{\psi}_*) = 1$ for all lifts of $\psi$ to finite covers of $\psi$, we have that some further nonzero power $\psi^{k'}$ of $\psi^k$ fixes a vector in $H_1(S, \mathbb{C}_\chi)$. Since there are $N$ irreducible representations of $\mathbb{Z}/N\mathbb{Z}$, it follows that there is a nonzero exponent $k_N$ such that the suspension $\widetilde{\psi}^{k_N}$ acting as a mapping class of $S_N$ has at least $N$ linearly independent fixed vectors. It follows that $T_{\widetilde{\psi}^{k_N}}$ has first Betti number at least $N+1$. Notice that $T_{\widetilde{\psi}^{k_N}}$ is a finite abelian cover of the manifold $T_{\psi^k}$. It follows that for each

$N$, the manifold $T_{\psi^k}$ has a finite abelian cover with first Betti number at least $N$. By Lemmas 5.2.1 and 5.2.2, it follows that $T_\psi$ is large. □

Theorem 1.5.1, has the following relatively easy corollary:

**Corollary 5.2.4.** *Let $T_\psi$ be a fibered hyperbolic 3–manifold with at least one cusp. Then $T_\psi$ is large.*

*Proof.* Let $S$ be a fiber of $T_\psi$. Then $S$ has at least one puncture. Choose a finite cover $S'$ of $S$ to which $\psi$ lifts and which has at least three punctures. Replacing the lift $\widetilde{\psi}$ by some positive power if necessary, we may assume that the punctures of $S'$ are preserved point–wise. Let $X$ denote suspension of the lift of $\widetilde{\psi}$ acting on $S'$.

Let $[\gamma]$ denote the homology class of a small loop about one of the punctures of $S'$. Clearly, $[\gamma]$ is invariant under the action of $\widetilde{\psi}$. There is a sequence of finite, abelian, $\widetilde{\psi}$–invariant covers of $S'$ which "unwind" $\gamma$. For each $N$, we can find such a cover $S_N$ such that the number of punctures is at least $N$. Replacing $\widetilde{\psi}$ by a further power will stabilize these punctures pointwise, resulting in a finite abelian cover of $X$ whose first Betti number is at least $N$. It follows that $X$ is large. □

Corollary 5.2.4 is a special case of a result of Cooper, Long and Reid in [9].

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